Lecture 8: Complete Languages

1 Introduction

The success of NP-completeness theory in unifying disparate areas of algorithmics naturally lead one to ask if the same techniques work for other natural classes such as P or NLOG.

Alternatively, the framework of the Basic Inclusion Lemma requires that we find complete languages for such classes to act as surrogates for the entire class.

We will show that for the canonical list DLOG, NLOG, PLOG, P, NP, PSPACE, DEXPT, NEXPT, etc, all but one class has complete languages. The exception for PLOG is not known to have complete languages, and the conjecture is that it does not have any.

Actually, we need to specify the reducibility in the definition of complete languages; it turns out all the usual reducibilities would work fine with usual caveat: we want the class to be closed under the chosen reducibility. For this reason, we will assume many-one logspace reducibility in these results. This choice is only too strong for the class DLOG for which a simple substitute is easily found.

2 P has a complete language

Fix $\Sigma = \{0,1\}$. Recall that there is an efficient universal acceptor $U = \{U_0, U_1, \ldots\}$ for the characteristic class $P|\Sigma$. We assume that U_i takes time $O(n^{|i|})$.

We define the following language over $\{0,1,\#\}$:

$$L^P = \{i\#^m x : i, x \in \Sigma^*, m = |x|^{|i|}, x \in L(U_i^P)\}.$$

We now show that L^P is P-complete. There are two facts to establish:

(1) L^P can be accepted in deterministic polynomial time. In fact, linear time suffices: on any input, we first check that it has the format $i\#^m x$ for some i, x, m. This is trivial. Then we check that m is correct: this takes time linear in the input length. If m is correct, we finally simulate U_i^P on input x. By assumption, this takes $\leq |x|^{|i|}$ time, which is linear in the input length.

(2) If $L' \in P$, then $L' \leq_m^L L^P$. Assume $L' \subseteq \Gamma^*$ and $h : \Gamma \to \{0,1\}^*$ is a fixed length encoding of symbols in Γ. Then $L' \in P$ implies $h(L') \in P$. Hence we may assume $h(L')$ is accepted by U_i^P for some fixed i. We construct a logspace transducer T: on input x, we output

$$i\#^m h(x)$$

where $m = |h(x)|^{|i|}$. This can be done is logspace. Clearly, $x \in L'$ iff $T(x) \in L^P$.

3 Natural Complete Languages

The above proof can be generalized to the other canonical classes, with the exception noted. But such complete languages cannot be said to be natural. "Naturalness" of complete languages are important in practice (not theoretically) – if we recognize a real world problem L that is closely related to one of these natural complete problems, we may suspect (and verify) if L is indeed complete too. This is the main contribution of the Karp follow-up to Cook's original demonstration that SAT is NP-complete.

We show: GAP is complete for $NLOG$.