Chapter 22

Teleportation

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Please attache the following notes to your chapter on Quantum Complexity.

–Chee

22.0.1 Teleportation

We give a surprising application of Bell states.[1] Suppose Alice and Bob met a long time ago to produce a Bell state such as $(|00\rangle + |11\rangle)/\sqrt{2}$ in a pair of qubits (named a and b). Then Alice left for a distant planet, carrying the a-qubit with her. Bob stayed behind on earth with the b-qubit. The important thing to remember is that as long as the two qubits remain isolated from the outside world, they remain "entangled" regardless of physical distance. For instance if Bob measures his b-qubit and sees a 0, then Alice's a-qubit will instantaneously turn into a 0. In some sense, this information has been transmitted faster than the speed of light. But this does not seem to do anything useful in real life. The discovery of teleportation by Bennett et al [?] shows how something new could be actually be achieved using such states.

One way to produce the Bell state is to begin with the ab-quword in state $|00\rangle$, subject a to the H-gate and then subject b to the control-NOT (*i.e.*, T_2) gate (using a as control line). See figure 22.1 (a).

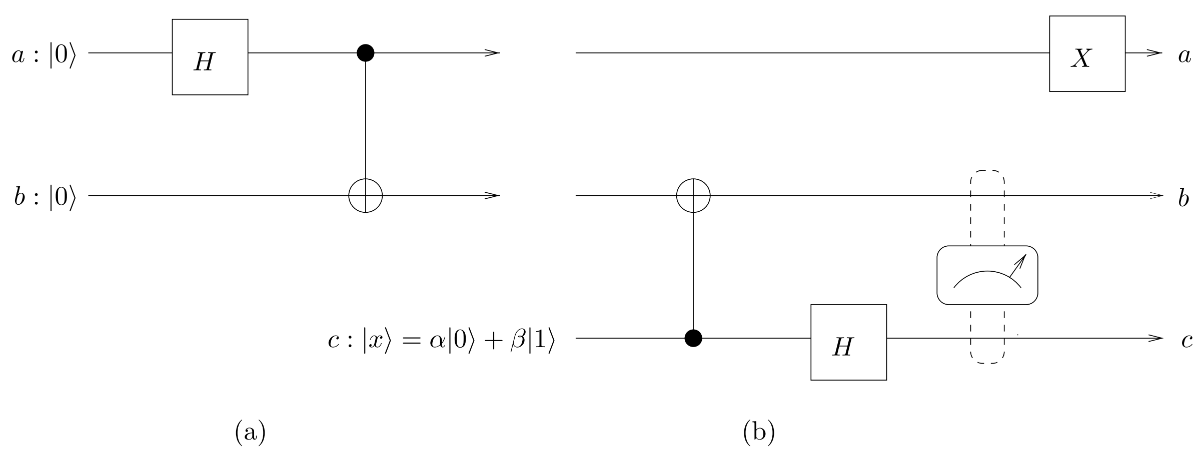

Figure 22.1: (a) Producing a Bell State, (b) Teleporting state $|x\rangle$.

Suppose Bob decided today to send Alice a quantum state $|x\rangle$ of a brand new qubit named c. This state was non-existent when Alice and Bob met long ago, and we might even assume that Bob know nothing about this state. Is there any way for Bob to convey this state to Alice, other than physically moving qubit c to the distant planet? We describe a protocol for transmitting $|x\rangle$ to Alice, by sending 2 classical bits using any classical channel, say, using radio communication. The idea is that Bob first entangles the b and c qubits, performs a measurement of b and c, and sends the results of this measurement (via radio) to Alice. Based on this information, Alice performs a ransformation of her a-qubit and behold, the state of the a qubit will be unknown $|x\rangle$.

[1]Named after John S. Bell (1964). Also called EPR pairs, after A. Einstein, N. Rosen and B. Podolsky (1935).

Here are the details, illustrated in Figure 22.1(b). Let $|x\rangle = \alpha|0\rangle + \beta|1\rangle$ for unknown $\alpha, \beta \in \mathbb{C}$. Then the state of the abc-quword is initially

$$\begin{aligned} |x_0\rangle &:= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= \frac{1}{\sqrt{2}}\{\alpha(|000\rangle + |110\rangle) + \beta(|001\rangle + |111\rangle)\}. \end{aligned}$$

- Bob first performs a control-NOT to the b-qubit (using the c-qubit as control). This produces the state

$$|x_1\rangle := \frac{1}{\sqrt{2}}\{\alpha(|000\rangle + |110\rangle) + \beta(|011\rangle + |101\rangle)\}.$$

- Then Bob subjects c to an H-gate, to produce the state

$$\begin{aligned} |x_2\rangle &:= \frac{1}{2}\{\alpha(|00\rangle + |11\rangle) \otimes (|0\rangle + |1\rangle) + \beta(|01\rangle + |10\rangle) \otimes (|0\rangle - |1\rangle)\} \\ &= \frac{1}{2}\{(\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle + (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\} \end{aligned}$$

  where we have grouped the a-states by each possibe outcome for the bc-qubits.

- Now Bob measures the bc-qubits (in Figure 22.1(b), this is indicated by the meter). This yields the bits $b_1c_1 \in \{00, 01, 10, 11\}l$, each choice occurring with probability 1/4. The a-qubit simultaneously collapses to the corresponding state, which is (respectively)

$$(\alpha|0\rangle + \beta|1\rangle)/\sqrt{2}, \qquad (\alpha|0\rangle - \beta|1\rangle)/\sqrt{2}, \qquad (\alpha|1\rangle + \beta|0\rangle)/\sqrt{2}, \qquad (\alpha|1\rangle - \beta|0\rangle)/\sqrt{2}$$

- Bob sends the two classical bits b_1c_1 to Alice using radio waves.
- Based on the received b_1c_1, Alice can apply a simple unitary transformation $X = X_{b_1c_1}$ to her a-qubit to recover the state $|x\rangle$. For instance, if $b_1c_1 = 00$, then the a-qubit is already in the required state.

Note that the requirement of a classical channel such as radio communication guarantees that we have not violated any fundamental principals such as transmitting information faster than the speed of light.

Bibliography

[1] P. Benioff. Quantum ballistic evolution in quantum mechanics: Applications to quantum computers. *Phy. Rev. A*, 54(2):1106–1123, 1996.

[2] E. Bernstein and U. Vazirani. Quantum complexity theory. *Proc. ACM Symposium on Theory of Computing*, 25:11–20, 1993.

[3] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca. Quantum algorithms revisited. *Proc. Royal Soc. London, A*, 454:339–354, 1998.

[4] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proc. Royal Soc. London, A*, 400(97–117), 1985.

[5] D. Deutsch and R. Jozsa. Rapid solution of problems by quantum computation. *Proc. Royal Soc. London, A*, 439(553–558), 1992.