Solution to HW2 of Honors Theory

March 1, 2002

1. (0 Point)

2. (10 + 15 Points)
   (i) We construct a NFA M with $n+1$ states that accepts L_n.

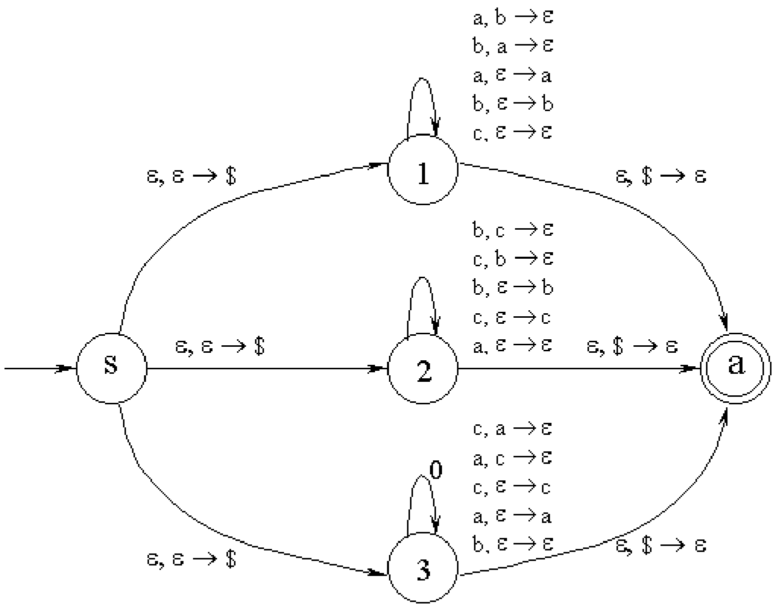
   $M = (Q, \Sigma, \delta, q_0, F)$
   $Q = \{q_0, q_1 ... q_n\}$
   $\Sigma = \{a_1, a_2 ... a_n\}$
   $F = \{q_1, q_2 ... q_n\}$
   $\delta(q_0, \epsilon) = q_i$, $i \neq 0$
   $\delta(q_i, a_j) = q_i$, $i \neq j$, $i \neq 0$.

   (ii) Assign to each string w a binary characteristic number $f(w) = b_n ... b_0$, where $b_i = 0$ if a_i does not appear in w, and $b_i = 1$ if a_i appears in w. There are 2^n such numbers. Only the strings that have the characteristic number with all bits set to 1 are rejected.
   Show that every DFA that accepts L_n has at least 2^n states.
   Suppose not, then there is a DFA M with less than 2^n states that accepts L_n. According to Pigeonhole Principle, there are at least two strings w and w' with different characteristic numbers x and y that reach the same state q_1. We have:
   $\delta(q_0, w) = q_1$
   $\delta(q_0, w') = q_1$.
   As $x \neq y$, they have at least one bit b_k that is different. WLOG, assume $b_k = 1$ in x, and $b_k = 0$ in y. Choose a string $s = a_1 ... a_{k-1} a_{k+1} ... a_n$, then $f(ws)$ has all bits equal to 1, and $f(w's)$ has all bits except b_k equal to 1. So ws should be rejected, while $w's$ should be accepted. But
   $\delta(q_0, ws) = \delta(q_1, s) = \delta(q_0, w's)$.
   We obtain a contradiction.

3. (15+10 Points)
   (i) As A is context free, there is a PDA $M_1 = \{Q_1, \Sigma_1, \Gamma_1, \delta_1, S, F_1\}$ that recognizes it. As B is regular, there is a DFA $M_2 = \{Q_2, \Sigma_2, \delta_2, q_0, F_2\}$ that recognizes it.
   We construct a PDA M that recognizes A/B:
   $M = \{Q, \Sigma, \Gamma, \delta, S, F\}$

$Q = Q_1 \cup (Q_1 \times Q_2)$
$\Sigma = \Sigma_1 + \Sigma_2$
$\Gamma = \Gamma_1$
$F = F_1 \times F_2$.
First copy all the states and transitions of M_1 to M, but make all the accepting states of M_1 nonaccepting in M. Then add a new state q_{ij} for each pair (q_i, q_j), where $q_i \in Q_1, q_j \in Q_2$. The accepting states are those q_{ij}'s, with $q_i \in F_1$, and $q_j \in F_2$.
Add transitions between the states in $Q_1 \times Q_2$:
for each symbol a such that $q_i, a, A \rightarrow q_I, B$ in M_1, and $q_j, a \rightarrow q_J$ in M_2 add a transition $q_{ij}, \epsilon, A \rightarrow q_{IJ}, B$
Add transitions between Q_1 and $Q_1 \times Q_2$:
for each $q_i \in Q_1$ add a transition $q_i, \epsilon, \epsilon \rightarrow q_{i0}, \epsilon$.
(ii)
$A = \{0^p 1^n, p \text{ is prime}, n \geq p\}$
$B = \{01^*\}$
$C = \{0^{p-1}, p \text{ is prime }\}$.
As B is regular, if A is context free, then $C = A/B$ is context free. But we will show that C is not context free, so A is not context free.
Use the pumping lemma to show that the language
$C = \{0^{p-1}, p \text{ is prime }\}$ is not context free.
We assume that C is context free and obtain a contradiction. Let l be the pumping length for C that is guaranteed to exist by the pumping lemma. Select the string $s = 0^{m-1}$ where m is a prime greater than l. Clearly s is a member of C and of length at least l. According to pumping lemma, we can pump s by dividing $s = uvxyz$, where $|vxy| \leq l$ and $|vy| > 0$.
Pick $i = m + 1$,
$|uv^ixy^iz| = |uvxyz| + |vy| \times (i-1) = m - 1 + m \times |vy| = m \times (|vy| + 1) - 1$
As $m \geq 2, |vy| \geq 2$, $|uv^ixy^iz| + 1$ can not be a prime. Thus s can not be pumped. So C is not a CFL.

4. (10+10+10 Points)
(i) L_1 is context free, as there is a PDA that recognizes it.


a, b → ε
b, a → ε
a, ε → a
b, ε → b
c, ε → ε
ε, ε → $
1
ε, $ → ε
b, c → ε
c, b → ε
b, ε → b
c, ε → c
a, ε → ε
s
ε, ε → $
2
ε, $ → ε
a
c, a → ε
a, c → ε
0
c, ε → c
a, ε → a
b, ε → ε
ε, ε → $
3
ε, $ → ε