Honors Theory of Computation, Spring'00 (Yap)
HOMEWORK 1:
OUT: Jan 19, DUE: Jan 31

QUESTION

Solve Problem 1.4.12 (Chapter 1, Papadimitriou's bk)
IMPORTANT: you must answer the entire question in one page (with reasonable size handwriting). When necessary, you must summarize the method or argument.
FOR YOUR CONVENIENCE, HERE IS THE QUESTION:
Show that, if in the augmentation algorithm of Section 1.2 we always augment by a shortest path, then (a) the distance of a node from s cannot decrease from an $N(f)$ to the next. Furthermore, (b) if edge (i, j) is the bottleneck at one stage (i.e., it is the edge along the augmenting path with the smallest capacity), then the distance of i from s must increase before it becomes a bottleneck in the opposite direction. (c) Conclude that there are at most $O(|E|.|V|)$ augmentations.

SOLUTION (a), 10 Points. The following argument is from Alexander Totok. It seems to be the shortest complete argument.

Let $N(f_i)$ be the derived network based on some flow f_i $(i = 1, 2)$. Let $d_i(u, v)$ denote the length of the shortest path from vertex u to vertex v in $N(f_i)$. For simplicity, write $d_i(v)$ for $d_i(s, v)$ where s is the source.

Suppose f_2 is derived from f_1 by augmentation along some path P_1. To prove part (a), suppose $d_2(v) < d_1(v)$ for some vertex v. We will derive a contradiction. Let $P_i(v)$ be a shortest path from s to v. Clearly, there is an edge (u', v') on $P_i(v)$ such that

$$d_2(u') \geq d_1(u'), \qquad d_2(v') < d_1(v').$$

Note that (u', v') cannot be in $N(f_1)$ because this would imply that

$$d_1(v') \leq d_1(u') + 1 \leq d_2(u') + 1 = d_2(v').$$

Hence (v', u') must be in the augmenting path P_1. This implies

$$d_1(u') = d_1(v') + 1 > d_2(v') + 1.$$

But we also have

$$d_2(v') = d_2(u') + 1 \geq d_1(u') + 1.$$

These 2 inequalities imply $d_1(u') > d_1(u') + 2$, a contradiction.

SOLUTION (b), 6 Points. Now, assume that f_2 is some flow that is obtained from f_1 after one or more augmentations along shortest paths. Suppose (i, j) is an edge of a shortest path from s to j in $N(f_1)$, and (j, i) is an edge of a shortest path from s to i in $N(f_2)$. Then

$$\begin{array}{rcll} d_2(i) & = & d_2(j) + 1 & \text{(since } (j,i) \text{ is on a shortest path of } N(f_2)) \\ & \geq & d_1(j) + 1 & \text{(from part (a))} \\ & = & d_1(i) + 2 & \text{(since } (i,j) \text{ is on a shortest path of } N(f_1)). \end{array}$$

In particular, if (i, j) is a bottle neck when we augment f_1 and (j, i) occurs on the augmentation path which leads to f_2, we obtain $d_2(i) \geq d_1(i) + 2$.

SOLUTION (c), 6 Points. Consider any edge (i, j) that is a bottleneck in some augmenting path. This means (i, j) disappears from the next derived network. For (i, j) to reappear, we need (j, i) to occur in some augmentation path. So part(b) implies that $d(i)$ would increase by at least 2 for (i, j) to reappear. Note that $i \neq s$ since this means that $(j, i) = (j, s)$ is used in a shortest path from s to $i = s$. Since $d_2(i) \leq |V| - 1$, we conclude that every edge (i, j) is a bottleneck for at most $|V|/2$ times. Hence, there are at most $|E|.|V|/2$ bottlenecks. As each augmentation has at least one bottleneck, this means there are at most $|E|.|V|/2$ augmentations.