G22.3033 Computational Geometry
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Homework #4

Problem 1: Let S be a finite set of spheres (or weighted points). Define the Voronoi diagram as usual.

(a) Give an explicit (numerical) example where a sphere $\hat{p} \in S$ is redundant. Recall that Edelsbrunner defines a point to be redundant if its Voronoi region is empty.

(b) Let $\hat{p} = (p, P^2)$. If p is a vertex of the convex hull of the centers in S, then $\hat{p}$ is not redundant.

Solution:

(a) Let S be a the following set of points:

$\hat{p}_0$	((1, 1, 1), 25)
$\hat{p}_1$	((1, 1, -1), 25)
$\hat{p}_2$	((1, -1, 1), 25)
$\hat{p}_3$	((1, -1, -1), 25)
$\hat{p}_4$	((-1, 1, 1), 25)
$\hat{p}_5$	((-1, 1, -1), 25)
$\hat{p}_6$	((-1, -1, 1), 25)
$\hat{p}_7$	((-1, -1, -1), 25)
$\hat{p}_8$	((0, 0, 0), 1)

$\hat{p}_0 \ldots \hat{p}_7$ are points of weight 25 on the verticies of an axis aligned cube that extends from -1 to 1 on each axis and $\hat{p}_8$ is a point of weight 1 at the origin. Consider any point $q = (x, y, z)$. Let $\hat{p}$ be the closest point among $\hat{p}_0 \ldots \hat{p}_7$ to q. The weighted distance function (or *power*) from $\hat{p}$ to q is as follows:

$$\begin{aligned} \pi_{\hat{p}}(q) &= ||q - p||^2 - P^2 \\ &= ((|x| - 1)^2 + (|y| - 1)^2 + (|z| - 1)^2) - 25 \end{aligned}$$

The weighted distance function from x to $\hat{p}_8$ is:

$$\begin{aligned} \pi_{\hat{p}_8}(q) &= ||q - p_8||^2 - P_8^2 \\ &= x^2 + y^2 + z^2 - 1 \end{aligned}$$

Now look at two cases. In the first case, $|x| \geq 1, |y| \geq 1, |z| \geq 1$. The following inequalities are trivially true:

$$\begin{aligned} (|x| - 1)^2 &< x^2 \\ (|y| - 1)^2 &< y^2 \\ (|z| - 1)^2 &< z^2 \\ -25 &< -1 \end{aligned}$$

Therefore $\pi_{\hat{p}}(x) < \pi_{\hat{p}_8}(x)$. Now examine the case where one or more of x, y, z is less than or equal to one. The following inequality is true because each term on the left hand side of the equation may be at most one larger than the corresponding term on the right hand side of the equation.

$$((|x| - 1)^2 + (|y| - 1)^2 + (|z| - 1)^2) \leq x^2 + y^2 + z^2 + 3$$

Substituting into the weighted distance equations:

$$\begin{aligned} ((|x| - 1)^2 + (|y| - 1)^2 + (|z| - 1)^2) - 25 &< x^2 + y^2 + z^2 - 1 \\ \pi_{\hat{p}}(x) &< \pi_{\hat{p}_8}(x) \end{aligned}$$

We now see that every point q is closer to one of $\hat{p}_0 \ldots \hat{p}_7$ than it is to $\hat{p}_8$. The Voronoi region of $\hat{p}_8$ is empty. $\hat{p}_8$ is redundant.

(b) LEMMA: Given two non-coincident weighted points:

$$\begin{aligned} \hat{p} &= (p, P^2) \\ \hat{q} &= (q, Q^2) \end{aligned}$$

$\overrightarrow{r} \in \mathbb{R}^3, r \cdot \overrightarrow{pq} \neq 0$, and a point s on $p + \overrightarrow{r}$ the difference:

$$\|s-p\|^2 - \|s-q\|^2$$

increases linearly with the distace $\|s-p\|^2$.

PROOF: In homework 3, exercise 3(a), we saw that this is **not** true for a line perpendicular to the segment connecting $\hat{p}$ and $\hat{q}$. We also saw that:

$$\|s-p\|^2 - \|s-q\|^2 = 2x(x_q - x_p) + 2y(y_q - y_p) + 2z(y_q - y_p)$$

Assuming this quantity is varying with $\|s-p\|^2$, it can only vary lineraly. Homework 3(b) showed that this quantity must vary if $\overrightarrow{r}$ is not perpendicular to $\overrightarrow{pq}$

□

LEMMA: Given the same two non-coincident weighted points as the previous lemma, and a half-line $p + \overrightarrow{r}$, there exists some finite distance R^2 such that all points on the half-line $p + \overrightarrow{r}$ at a distance of R^2 or greater from p are closer to the same point (p or q) under both the standard distance function and the weighted distance function.

PROOF: Assume $\|s-p\|^2 - \|s-q\|^2$ is increasing linearly with R^2. At some radius, $\|s-p\|^2 - \|s-q\|^2$ will equal than $P^2 - Q^2$. Greater than this radius, $\|s-p\|^2 - \|s-q\|^2$ will be positive iff $\|s-p\|^2 - \|s-q\|^2 - P^2 + Q^2$ is positive. This means that the distance from $\hat{p}$ is the same under both distance measures.

□

Assume S is non-degenerate. Let $\hat{p} = (p, P^2)$ be a point where p is on the convex hull of the centers of S. Let v be a half line starting at p and heading toward infinity located entirely within the Voronoi region of p. All points on v are closer to p than they are to any other point in the centers of S. By the Lemma above, there is some R_i^2 where for each p_i such that points on v are closer to $\hat{p}$ than they are to $\hat{p}_i$ under the weighted distance measure. All points on v at a radius of $\max(R_i)$ or greater are in the weighted Voronoi region of $\hat{p}$. $\hat{p}$'s weighted Voronoi cell is not empty. Therefore it is not redundant.

Problem 2: From the problems on the homework sheet.

(a) **(PROBLEM 1)** Show that $t(n) = \Omega(n^2)$. HINT: Give a tetrahedralization of S_n with $t(K) = \Omega(n^2)$.

(b) **(PROBLEM 2)** Is it true that every tetrahedralization of S_n has $t(K) \geq \Omega(n^2)$?

(c) **(PROBLEM 3)** Prove the relationships from Lemma 1.

(d) **(PROBLEM 5)** Show that the Delaunay triangulation of S_5 does not satisfy the maxmin solid angle criterion.

Solution:

(a) Label the points as follows: Points on the circle in the x-y plane are labeled $C_0, C_1, \ldots, C_{n/2}$. Points on the z-axis are labeled $A_0, A_1, \ldots, A_{n/2}$. The following is a tetrahedralization:

A_0, A_1, C_0, C_1
A_0, A_1, C_1, C_2
...
$A_0, A_1, C_{n/2}, C_0$
A_1, A_2, C_0, C_1
A_1, A_2, C_1, C_2
...
$A_{n/2-1}, A_{n/2}, C_{n/2-1}, C_{n/2}$
$A_{n/2-1}, A_{n/2}, C_{n/2}, C_1$

The total number of tetrahedra in this tetrahedralization is:

$$t(n) = \left(\frac{n}{2}\right)\left(\frac{n}{2} - 1\right)$$

$t(n) = \Omega(n^2)$.

(b) Begin by observing Lemma 1(e). v^B is obviously $n/2 + 2$ and v^I is obviously $n/2 - 2$. Now lets look at e^I. All of the verticies on the z-axis are colinear. No three of them may be part of the same tetrahedron. This means there must be at "column" of $n/2 - 1$ interior edges down the z-axis. All of the verticies in the x-y plane are coplanar. No 4 of them may be part of the same tetrahedron. Any tetrahedron therefore, must contain at least one edge from the column on the z-axis, and one edge from the circle on the x-y plane. Now, in order to fill all of the space in the convex hull of S, we must have $(n/2 - 2) * (n/2)$ internal edges – one from each internal point on the z axis to a point on the cicle. $t(n) = \Omega(n^2)$.

(c) (a) Induction on t. By inspection, the relation holds when $t = 1$. Assume that the relation holds for a certain t, and look at what happens when we begin adding tetrahedra. If the new tetrahedron shares one face with the existing tetrahedra, $f^I \to f^I + 1$, $f^B \to f^B + 3 - 1 = f^B + 2$, $e^I \to e^I$, $e^B \to e^B + 3$, $v^I \to v^I$, $v^B \to v^B + 1$, and $t \to t + 1$.

$$\begin{aligned} 4t &= f^B + 2f^I \\ 4(t+1) &= (f^B + 3 - 1) + 2(f^I + 1) \\ 4t + 4 &= f^B + 2f^I + 4 \end{aligned}$$

The relation still holds. If the new tetrahedron shares two faces with the existing tetrahedra, $f^I \to f^I + 2$, $f^B \to f^B + 2 - 2 = f^B$, $e^I \to e^I + 1$, $e^B \to e^B + 1 - 1 = e^B$, $v^I \to v^I$, $v^B \to v^B$, and $t \to t+1$.

$$\begin{aligned} 4t &= f^B + 2f^I \\ 4(t+1) &= (f^B + 2 - 2) + 2(f^I + 2) \\ 4t + 4 &= f^B + 2f^I + 4 \end{aligned}$$

The relation still holds. If the new tetrahedron shares three faces with the existing tetrahedra, $f^I \to f^I + 3$, $f^B \to f^B + 1 - 3 = f^B - 2$, $e^I \to e^I + 3$, $e^B \to e^B - 3$, $v^I \to v^I + 1$, $v^B \to v^B - 1$, and $t \to t+1$.

$$\begin{aligned} 4t &= f^B + 2f^I \\ 4(t+1) &= (f^B + 1 - 3) + 2(f^I + 3) \\ 4t + 4 &= f^B + 2f^I + 4 \end{aligned}$$

The relation still holds. Finally, If the new tetrahedron shares four faces with the existing tetrahedra, $f^I \to f^I + 4$, $f^B \to f^B - 4$, $e^I \to e^I + 6$, $e^B \to e^B - 6$, $v^I \to v^I + 4$, $v^I \to v^I - 4$, and $t \to t+1$.

$$\begin{aligned} 4t &= f^B + 2f^I \\ 4(t+1) &= (f^B + 1 - 4) + 2(f^I + 4) \\ 4t + 4 &= f^B + 2f^I + 4 \end{aligned}$$

These are the four possible ways to add a connected tetrahedron to an existing tetrahedralization, therefore this relation holds for all $t \geq 1$.

(b) Assume the relationship $f^B = 2v^B - 4$ holds for a simplicial complex K. We will show three transformations (all removing tetrahedra) which maintain this relationship.

Removing a tetrahedron with three boundary faces and one internal face: $f^I \to f^I - 1$, $f^B \to f^B - 3 + 1 = f^B - 2$, $e^I \to e^I$, $e^B \to e^B - 3$, $v^I \to v^I$, $v^B \to v^B - 1$, and $t \to t-1$.

$$\begin{aligned} f^B &= 2v^B - 4 \\ f^B - 3 + 1 &= 2(v^B - 1) - 4 \\ f^B - 2 &= (2v^B - 4) - 2 \end{aligned}$$

Removing a tetrahedron with two boundary faces and two internal faces: $f^I \to f^I - 2$, $f^B \to f^B - 2 + 2 - f^B$, $e^I \to e^I - 2$, $e^B \to e^B - 1 + 1 = e^B$, $v^I \to v^I$, $v^B \to v^B$, and $t \to t-1$.

$$\begin{aligned} f^B &= 2v^B - 4 \\ f^B + 2 - 2 &= 2(v^B + 0) - 4 \\ f^B &= 2v^B - 4 \end{aligned}$$

Removing a tetrahedron with one boundary face and three internal faces: $f^I \to f^I - 3$, $f^B \to f^B - 1 + 3 = f^B + 2$, $e^I \to e^I - 3$, $e^B \to e^B + 3$, $v^I \to v^I - 1$, $v^B \to v^B + 1$, and $t \to t-1$.

$$\begin{aligned} f^B &= 2v^B - 4 \\ f^B + 3 - 1 &= 2(v^B + 1) - 4 \\ f^B + 2 &= (2v^B - 4) + 2 \end{aligned}$$

Now, starting with K, remove tetrahedra with these transformations until only one tetrahedron remains. The relationship ($f^B = 2v^B - 4$) is observed to be true for this case, therefore it is true for all of the previous cases.

(c) Assume the relationship $f^I = v^B + 2(e^I - v^I) - 4$ holds true for a simplicial complex K. We will show three transformations (all removing tetrahedra) which maintain this relationship.

Removing a tetrahedron with three boundary faces and one internal face: $f^I \to f^I - 1$, $f^B \to f^B - 3 + 1 = f^B - 2$, $e^I \to e^I$, $e^B \to e^B - 3$, $v^I \to v^I$, $v^B \to v^B - 1$, and $t \to t - 1$.

$$\begin{aligned} f^I &= v^B + 2(e^I - v^I) - 4 \\ F^I - 1 &= (v^B - 1) + 2(e^I - v^I) - 4 \\ &= (v^B + 2(e^I - v^I) - 4) - 1 \end{aligned}$$

Removing a tetrahdedron with two boundary faces and two internal faces: $f^I \to f^I - 2$, $f^B \to f^B - 2 + 2 - f^B$, $e^I \to e^I - 2$, $e^B \to e^B - 1 + 1 = e^B$, $v^I \to v^I$, $v^B \to v^B$, and $t \to t - 1$.

$$\begin{aligned} f^I &= v^B + 2(e^I - v^I) - 4 \\ f^I - 2 &= v^B + 2((e^I - 1) - v^I) - 4 \\ f^I - 2 &= v^B + 2(e^I - v^I) - 4 - 2 \end{aligned}$$

Removing a tetrahedron with one boundary face and three internal faces: $f^I \to f^I - 3$, $f^B \to f^B - 1 + 3 = f^B + 2$, $e^I \to e^I - 3$, $e^B \to e^B + 3$, $v^I \to v^I - 1$, $v^B \to v^B + 1$, and $t \to t - 1$.

$$\begin{aligned} f^I &= v^B + 2(e^I - v^I) - 4 \\ f^I - 3 &= (v^B + 1) + 2((e^I - 3) - (v^I - 1)) - 4 \\ f^I - 3 &= (v^B + 2(e^I - v^I) - 4) + 1 - 6 + 2 \\ f^I - 3 &= (v^B + 2(e^I - v^I) - 4) - 3 \end{aligned}$$

From here the argument is the same as in part 2(c)(b).

(d) Induction on the number of tetrahedra t. A single tetrahedron has 6 boundary edges and 4 boundary faces. The relationship $2e^B = 3f^B$ is true by inspection. Now we add tetrahedra as in part 2(c)(a).

Add a tetrahedron with one internal face and three external faces: $f^I \to f^I + 1$, $f^B \to f^B + 3 - 1 = f^B + 2$, $e^I \to e^I$, $e^B \to e^B + 3$, $v^I tov^I$, $v^B \to v^B + 1$, and $t \to t + 1$.

$$\begin{aligned} 2e^B &= 3f^B \\ 2(e^B + 3) &= 3(f^B + 2) \\ 2e^B + 6 &= 3f^B + 6 \end{aligned}$$

Add a tetrahedron with two internal faces and two external faces: $f^I \to f^I + 2$, $f^B \to f^B + 2 - 2 = f^B$, $e^I \to e^I + 1$, $e^B \to e^B + 1 - 1 = e^B$, $v^I \to v^I$, $v^B \to v^B$, and $t \to t + 1$.

$$\begin{aligned} 2e^B &= 3f^B \\ 2(e^B + 1 - 1) &= 3(f^B + 2 - 2) \\ 2e^B &= 3f^B \end{aligned}$$

Add a tetrahedron with three internal faces and one external face: $f^I \to f^I + 3$, $f^B \to f^B + 1 - 3 = f^B - 2$, $e^I \to e^I + 3$, $e^B \to e^B - 3$, $v^I \to v^I + 1$, $v^B \to v^B - 1$, and $t \to t + 1$.

$$\begin{aligned} 2e^B &= 3f^B \\ 2(e^B - 3) &= 3(f^B + 1 - 3) \\ 2e^B - 6 &= 3f^B - 6 \end{aligned}$$

Add a tetrahedron with four internal faces: $f^I \to f^I + 4$, $f^B \to f^B - 4$, $e^I \to e^I + 6$, $e^B \to e^B - 6$, $v^I \to v^I + 4$, $v^I \to v^I - 4$, and $t \to t + 1$.

$$\begin{aligned} 2e^B &= 3f^B \\ 2(e^B - 6) &= 3(f^B - 4) \\ 2e^B - 12 &= 3f^B - 12 \end{aligned}$$

These are the four possible ways to add a connected tetrahedron to an existing tetrahedralization, therefore this relation holds for all $t \geq 1$.

(e) Start with (a):

$$\begin{aligned} 4t &= f^B + 2f^I \\ t &= \frac{f^B}{4} + \frac{f^I}{2} \end{aligned}$$

Substitute (b) and (c) into this equation:

$$\begin{aligned} t &= \frac{2v^B - 4}{4} + \frac{v^B + 2(e^I - v^I) - 4}{2} \\ &= \frac{v^B}{2} - 1 + \frac{v^B}{2} + e^I - v^I - 2 \\ &= v^B + e^I - v^I - 3 \end{aligned}$$

(d) Start with the triangulation: *abcd*, *abce*. The circumcirle of *abcd* is $((1, 1, 5/8), 1.54616^2)$. The circumcircle of *abce* is $((1, 1, 5/4), 1.88745^2)$. The distance from the point d to the circumcircle of *abce* is $.5^2 + .5^2 + .75^2 = 1.0625$. We must flip triangle *abc* to obtain a Delaunay triangulation.

Let (x, yzw) denote the solid angle between the point x and the triangle yzw. We can get the following intuition:

In the first triangulation, $(d, abc) < (e, abc)$. Why? d is farther from *abc* than e is, and both points are at the same angle to *abc*.

$(d, abe) + (d, ace) + (d, bce) = (d, abc)$. This is a geometric intuition we gain from “splitting” *abc*.

$(a, bce) + (a, bcd) = (a, bde) + (a, cde)$, $(b, acd) + (b, ace) = (b, ade) + (b, cde)$, and $(c, abd) + (c, abe) = (c, ade) + (c, bde)$. This is also geometric intuition from splitting *abc*.

$(d, abe) < (c, abe)$. Why?