Geometric Modeling: HW 4

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Question 1

Part a)

Consider the set of 3 circles $\hat{v} = ((0,0),0), \hat{u}_1 = ((-1,0),2^2)$ and $\hat{u}_2 = ((1,0),2^2)$. Consult Fig. 1 for a sketch. Notice that the "circle" $\hat{v}$ is, in fact, just the point v. We claim that no point of $\mathbb{R}^2$ belongs to $\hat{v}$'s voronoi cell.

Now, consider a point of $\mathbb{R}^2$ that lies in $\hat{u}_1 \cup \hat{u}_2$ (like p_1). Then for at least one of the $\hat{u}_i$'s, $\pi_{u_i}(p) < 0$. Since $\hat{v}$ has radius 0, for all points $q \in \mathbb{R}^2$ we have that $\pi_v(q) \geq 0$. Thus none of these points belong to v's voronoi region. In particular, this include the point v itself.

Next, consider a point of the form $p = (0, y)$ like p_2 in the figure. Then $\pi_v(p) = y^2$. We also have that $\pi_{u_1}(p) = \|(0,y) - (-1,0)\|^2 - 4 = y^2 - 3$ which is certainly less than y^2.

Finally, let $p = (x, y)$ with $p \notin \hat{u}_1$ and $p \notin \hat{u}_2$. Further, assume with loss of generality that $x > 0$.

$$\begin{aligned}\pi_{u_2}(p) &= \|(x,y) - (1,0)\|^2 - 4 \\ &= x^2 + y^2 - 2x - 3 \\ &< x^2 + y^2 \\ &= \pi_v(p)\end{aligned}$$

Part b)

If v lies on the convex hull of centers of the circles, then the normal cone at v is non-empty. Let w be the *vector* that maximizes the minimum dot product value between itself and any

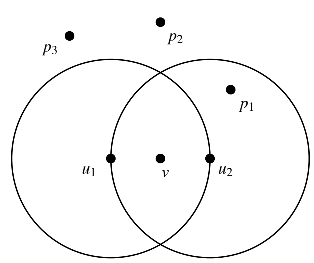

Figure 1: The p_i's correspond to the 3 combinatorial cases for points.

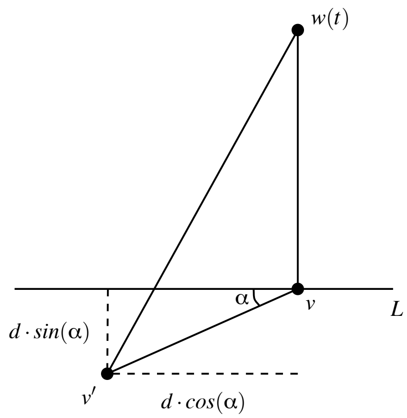

Figure 2:

vector formed by v and a neighboring voronoi center. Basically, I just want the "middle" vector from the normal cone. Assuming that w is of unit length, I define $w(t)$ to be $v + t * w$. Explicitly, it is the point of distance t from v in the direction of w.

By definition of w, we can draw a line L, perpendicular to w, such that all other circles centers lie below L with respect to $w(t)$. This assumes that v is not a convex combination of other vertices of the hull, otherwise, other points could lie on L.

Examine what happens as we increase the value of t. With respect to v we have that $\pi_v(w(t)) = \|w(t) - v\|^2 - V^2 = t^2 - V^2$. This means that

$$\begin{aligned}
\pi_v(w(t+1)) - \pi_v(w(t)) &= (\|w(t) - v\|^2 - V^2) - (\|w(t) - v\|^2 - V^2) \\
&= (t+1)^2 - t^2 \\
&= 2t + 1
\end{aligned}$$

This value is the increase in $\pi_v(\cdot)$ when we move one unit. Now we calculate the same for any other circle center, say v'. Since my latex alignment skills are quite poor, let's do some calculations so it doesn't get too long later.

$$\begin{aligned}
\pi_{v'}(w(t)) &= \|w(t) - v'\| - V'^2 \\
&= (t + d \cdot sin(\alpha))^2 + d \cdot cos(\alpha)^2 - V'^2 \\
&= t^2 + 2td \cdot sin(\alpha) + d - V'^2
\end{aligned}$$

The change is therefore

$$\pi_{v'}(w(t+1)) - \pi_{v'}(w(t)) = 2t + 1 + 2d \cdot sin(\alpha)$$

So we see that as t is increased, the weighted distance from $w(t)$ to v' grows faster than that of $w(t)$ to v. Thus we can keep increasing t until we reach a value t^* for which $\pi_v(w(t^*)) < \pi_{v'}(w(t^*))$. Since v' was arbitrary, we can do this for any other circle center as well. If we move beyond the maximal such t^* then we have a point that lies in v's voronoi region.

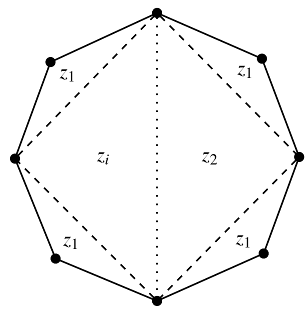

Figure 3: View of the xy-plane from the positive z side. The labels inside the triangles correspond to the vertex with which that triangle is paired to make a tetrahedron. The z_i refers to all remaining z points.

Question 2

For the sake of sanity label the z-axis points $z_1, z_2, \ldots, z_{n/4}$ in increasing z value. Consider the following tetrahedralization: start z_1 and z_2. Create tetrahedra with those two points and every two adjacent points in the xy-plane. This creates $n/2 - 1$ tetrahedra. Do the same with z_2 and z_3. This will create $(n/2-1)(n/4-1) = \frac{n^2}{8} - \frac{3n}{4} + 1$ tetrahedra. We can do the same with the negative points on the z-axis for a total of $\frac{n^2}{4} - \frac{3n}{2} + 2$ tetrahedra which is $O(n^2)$ tetrahedra.

Question 3

The answer is no. We change our notation and label the z-axis points $z_1, z_2, \ldots, z_{n/4}$ in *decreasing* z coordinate value. Now consider the following: starting from z_1, we create tetrahedra with z_1 and as many consecutive non-overlapping (except at endpoints) triplets of xy-plane points as possible. Consult figure 3 for a view of the xy-plane during this process. What remains now are the points $z_2, \ldots, z_{n/4}$ and $\lceil \frac{n}{2}/2 \rceil$ points in the xy-plane. We continue the same process for z_2 and so on. This process is always safe since the points in the xy-plane are in convex position and we only take "ears" of their convex hull.

We stop when there are three points left. This happens after at most $\log n - 2$ levels. If there are any z_i left, we tetrahedralize as in question 2 (pair consecutive z_i's with the 3 remaining points in the xy-plane). This means at most 3 tetrahedra for each remaining z_i point.

How many tetrahedra does this have? We have an upper bound of $\lceil n/4 \rceil + \lceil n/8 \rceil + \ldots + \lceil n/2^{\log n - 1} \rceil$ which is certainly less than $n + \log n - 1$ which is $O(n)$. The remaining z_i, if any, will be tetrahedralized with $3(n - \log n + 1)$ tetrahedra which is also $O(n)$. We can do the same to the bottom so that the entire tetrahedralization is also linear.

Question 4

a) $4t = f^B + 2f^I$. Notice that every tetrahedron has four faces, so the left hand side counts the number of incidences of a face to a tetrahedron. Each face is either internal to K or on the boundary. If it is internal, it is incident to two tetrahedra which is counted by $2f^I$. If it is on the boundary, then it is only incident to one; the other term counts this. This is equivalent to the Handshaking Lemma for graphs but with simplicial complexes.

b) Since the terms only involve the boundary, we can consider the boundary as a graph on a sphere. The explicit transformation can be done by surrounding the complex with a ball and choosing an internal point of the complex and ray shooting all vertices to the surface of the ball maintaining adjacency.

Since we have a graph on a sphere, we invoke Euler's formula $v - e + f = 2$. Further, from part d) we know that $3f = 2e$ in this graph (all faces are triangular). By substitution we have that $v - (\frac{3f}{2}) + f = 2$ which implies that $f = 2v - 4$.

c) We again use Euler's formula, this time on complexes equivalent to the 3-dimensional ball which gives $v - e + f - t = 1$. We first rearrange the equation then we use d), a) and finally b)

$$\begin{aligned}
f^I &= -v + e - f^B + t + 1 \\
&= -v + e^I + f^B/2 + t + 1 \\
&= -v + e^I + f^B/2 + f^B/4 + f^I/2 + 1 \\
&= -2v + 2e^I + \frac{3f^B}{2} + 2 \\
&= v^B - 2v^I + 2e^I - 4
\end{aligned}$$

d) Much like the argument for a), $3f^B$ counts the number of face-edge incidences from the face point of view. Each boundary edge contributes to two faces, so $2e^B$ counts the edge-face incidence. Thus, we have that $3f^B = 2e^B$.

e) Using the formulas from a), b) and c) we have that

$$\begin{aligned}
t &= \frac{f^B}{4} + \frac{f^I}{2} \\
&= \frac{2v^B - 4}{4} + \frac{v^B + 2(e^I - v^I) - 4}{2} \\
&= v^B - v^I + e^I - 3
\end{aligned}$$

Question 5

We denote the circumsphere of the point set $\{a, b, c, d\}$ by $o(a, b, c, d)$. For this question, we focus on the two tetrahedralizations in the figure. The left uses the tetrahedra $T_1 = \{abcd, abce\}$ and the second consists of $T_2\{abde, acde, bcde\}$. First, we show that T_1 is

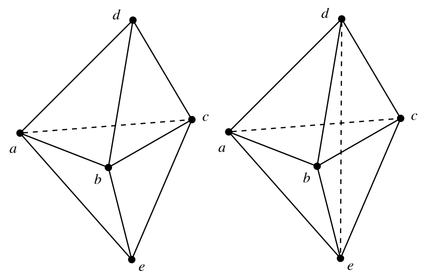

Figure 4: Two tetrahedralizations. T_1 has 2 tetrahedra, T_2 has 3

not the Delaunay one. Using the determinant formulas for circumspheres, we have that $o(a, b, c, d)$ is defined by $(x-1)^2 + (y-1)^2 + (z-0.625)^2 = 2.391$. Plugging in the point $e = (1.5, 0.5, -0.5)$ we have that

$$\begin{aligned}(1.5-1)^2 + (0.5-1)^2 + (-0.5-0.625)^2 &= 0.25 + 0.25 + 1.265 \\ &= 1.765 \\ &< 2.391\end{aligned}$$

This means that T_1 is not Delaunay since it does not satisfy the empty circumspheres property.

Moving on to T_2 we have the following equations for the tetrahedra

$$\begin{aligned}o(abde) &\to (x-1)^2 + (y-0.5)^2 + (z-0.75)^2 = 1.813 \\ o(acde) &\to (x-0.75)^2 + (y-1.25)^2 + (z-0.75)^2 = 2.688 \\ o(bcde) &\to (x-1.5)^2 + (y-1)^2 + (z-0.75)^2 = 1.813\end{aligned}$$

For each tetrahedron, we plug in the point that does not belong to it

$$\begin{aligned}o(abde) \ \&\ c &\to (2-1)^2 + (2-0.5)^2 + (0-0.75)^2 = 3.813 > 1.813 \\ o(acde) \ \&\ b &\to (2-0.75)^2 + (0-1.25)^2 + (0-0.75)^2 = 3.688 > 2.688 \\ o(bcde) \ \&\ a &\to (0-1.5)^2 + (0-1)^2 + (0-0.75)^2 = 3.813 > 1.813\end{aligned}$$

We conclude that all the circumspheres of the tetrahedra are empty of other points of the set. So the latter is the Delaunay tetrahedralization.

Moving on to the maxmin solid angle property, we have the minimum solid angle of 0.1229 steradians in T_1 achieved at the vertices a and c in the tetrahedron $\{bcde\}$. In T_2, the solid angle at d in tetrahedron $\{abde\}$ is only 0.1095 steradians. Thus T_2 does not satisfy the maxmin solid angle property.