Homework 3

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Question 1

For this question, from the class notes, we see that we need only calculate the additional sign information in the form of the σ_i's. Thus we can alter Collins' algorithm with a variable that maintains the current $\prod_{i=1}^{j} s_{2i-1}$ information as well as another to store information for the odd case. Updating takes constant time since we need only multiply by the new s_k of the current iteration.

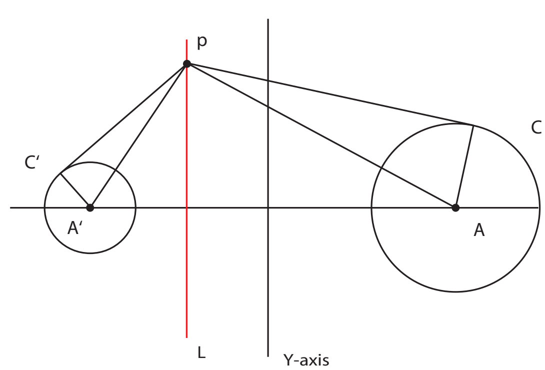
The number of operations is constant on each iteration and so doesn't not change the complexity. For the bit complexity, we note that the σ_i's just range over the $sign(x)$ operation and so don't blow up at all. So the bit complexity is also unchanged.

Question 2

If we only want the signs then we need only store the leading term of the polynomials at each step. This saves the computation required in the $prem(\cdot,\cdot)$ to only $O(m)$. The rest of the Collins' algorithm, specifically steps 4 and 5 are unchanged since these steps are not effected since they already involve only the first terms of the polynomials. Still, this reduced the complexity by a linear factor to $O(m^2)$.

Question 3

Part a)



Considering the case as in the question with A on the positive side and A' on the negative of the x-axis and letting our point p be (x, y), we see that

$$pow(p, C) = ((A - x)^2 + y^2) - r^2$$
$$pow(p, C') = ((x - A')^2 + y^2) - r'^2$$

Subtracting, we get $pow(p, C) - pow(p, C') = (A - x)^2 - (x - A')^2$. This term in no way involves the coordinate y. Since x was arbitrary, we conclude that along any vertical line, the term $pow(p, C) - pow(p, C')$ is constant.

Part b)

Assume that p_0 is the point in question with $pow(p_0, C) = pow(p_0, C')$. Along the x-axis we have that $y = 0$ so we have that

$$pow(p, C) = (A - x)^2 - r^2$$
$$pow(p, C') = (x - A')^2 - r'^2$$

Setting them to be equal, we have that $A^2 - 2Ax + x^2 - r^2 = A'^2 - 2A'x + x^2 - r'^2$. We can solve for x with

$$x = \frac{(A^2 - r^2) - (A'^2 - r'^2)}{2(A - A')}$$

which is is well-defined since $A' < 0$ so the denominator is always non-zero. Also, this is unique.

Consider "moving left" of p_0. That is, consider staying on the x-axis but with $x = p_0 - \epsilon$. Then

$$\begin{aligned} pow(x, C) &= A^2 - 2A(p_0 - \epsilon) + (p_0 - \epsilon)^2 - r^2 \\ &= pow(p_0, C) + 2A\epsilon - 2p_0\epsilon + \epsilon^2 \end{aligned}$$

Similarly for C' we have

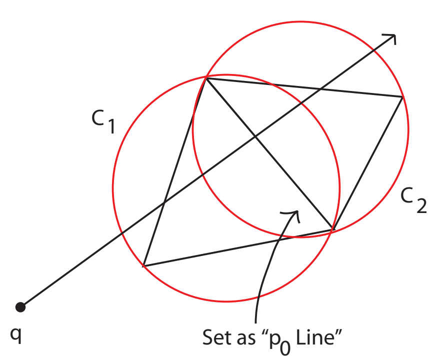
$$pow(x, C') = pow(p_0, C') + 2A'\epsilon - 2p_0\epsilon + \epsilon^2$$

The difference is in the terms of $2A\epsilon$ and $2A'\epsilon$. Since $A' < 0$, we conclude that $pow(x, C') < pow(x, C)$. Similarly, "moving right", makes $pow(x, C) < pow(x, C')$.

To help with our analysis, we recall that the difference between powers does not change along vertical lines. Thus, if $pow(p, C) > pow(p, C')$ then this remains true along the horizontal line through p with respect to the circle centers.

Now, let p be *any* point left of p_0 (not just along the x-axis). Then p has a horizontal line through it that intersects the x-axis, say p_x. From b) we can conclude then that $pow(p_x, C') < pow(p_x, C)$ and so too can we conclude that $pow(p, C') < pow(p, C)$

Part c)



Consider a sequence of triangles from a fixed viewpoint q as in Edelsbrunner's book. Then for any two sequential triangles and their circumcircles C_1, C_2, they share an edge that is perpendicular to the line formed by the two circle centers (explicitly, the Delaunay edge). We recall that this Delaunay edge is a subset of the points that are equidistant from the circle centers thus all points along this line have $pow(p_0, C_1) = pow(p_0, C_2)$. Then from part b) we can also conclude that $pow(p_0, C_1) < pow(p_0, C_2)$. By an inductive argument, we can conclude Edelsbrunner's acyclicity theorem.