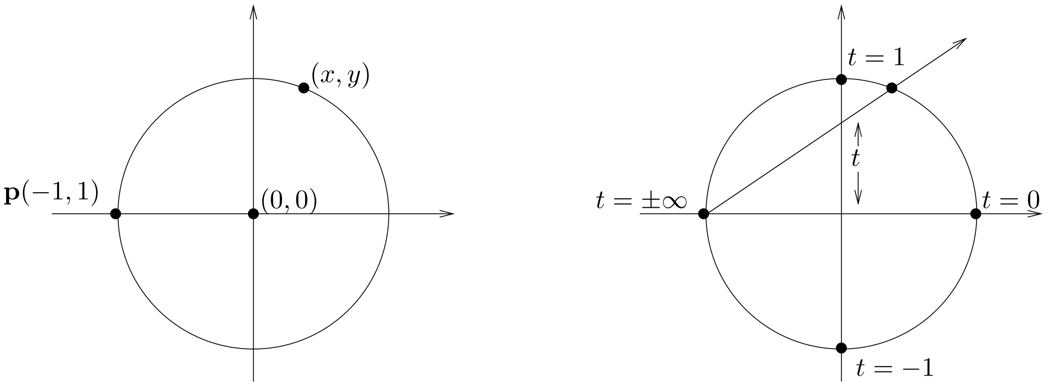
Lecture I
CURVES

What is a curve? Euclid proposed three descriptions (see [1]). The answer is not obvious, even in the plane. We might say a curve is a continuous function $\gamma : [0, 1] \to \mathbb{R}^2$. But this may convey more information that we need – the parametrization of points on this curve may not be part of our notion of a curve. We can view the the curve simply as the image of γ. But for other purposes, this may convey too little information. In particular, we need some effective or constructive methods of describing the curve. We can thus define a curve in $\mathbb{R}^2$ as the set of points that satisfy a given equation $f(x, y) = 0$. Effectivity is achieved, for instance, if the function $f(x, y)$ is a polynomial with integer coefficients. This is essentially the view point of algebraic geometry. From a differential geometry viewpoint, notions of effectivity might involve the the curvature, etc. Finally in the fielf of geometric modeling, the notion of effectivity is stronger – we want curves that are "controllable". In some sense, any parametric family of curves is controllable (e.g., we can control the algebraic curves which are defined by polynomials by changing their coefficients). Controllability in geometric modeling refers to parameters that bears a more direct relation to the actual spatial layout of the curve. This brings us to topics such as splines, Bézier curves and NURBS. This will be the main focus of this chapter.

§1. Implicit and Parametric Curves

Let $f(x, y)$ be a polynomial with real coefficients. The equation $f(x, y) = 0$ defines a curve in $\mathbb{R}^2$. For instance, $f(x, y) = x^2 + y^2 - 1$ defines the unit circle. See figure 1(a).



(a) $x^2 + y^2 - 1 = 0$ (b) $(x, y, w) = (1 - t^2, 2t, 1 + t^2)$

Figure 1: Unit Circle

On the other hand, we can also represent the circle as a curve $\mathbf{p}(t) = (x(t), y(t))$ where

$$x(t) = \frac{1 - t^2}{1 + t^2}, \qquad y(t) = \frac{2t}{1 + t^2}.$$

We call $x(t), y(t)$ the **coordinate functions** of the curve. Thus $\mathbf{p}(0) = (1, 0)$ and $\mathbf{p}(1) = (0, 1)$. This is illustrated in figure 1(b). Note the the point $(-1, 0)$ is represented twice by this representation: $\mathbf{p}(\infty) = \mathbf{p}(-\infty) = (-1, 0)$. Alternative, in homogenous coordinates, the curve can be written as

$$(x(t), y(t), w(t)) = (1 - t^2, 2t, 1 + t^2).$$

This simple example illustrates two ways to represent a curve: as an equation (or system of equations in higher dimensions) or in terms of its coordinate functions. They are called, respectively, the **implicit representation** and **parametric representaion** of curves. We restrict the functions in both representations to be rational (i.e., a ratio of two polynomials). Depending on the applications, one or the other may be better. Implicit representations are more general: we will see that every parametric representation has an implicit form but not vice-versa. Parametric representations are very useful when we need to perform "curve tracing", namely to plot points on the curve to be used in visualization or rendering.

§2. Casteljau's Algorithm and Bézier Curves

Consider curves in Euclidean space $\mathbb{R}^d$, $d \geq 1$. A **polynomial parametric curve** in $\mathbb{R}^d$ is a function $\mathbf{f} : [u,v] \to \mathbb{R}^d$ such that $\mathbf{f}(t) = \sum_{j=0}^{n} \mathbf{p}_i t^i$ for some $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^d$, and $-\infty \leq u \leq v \leq +\infty$. The domain of $\mathbf{f}$ is typically $[u,v] = [0,1]$ but sometimes $[u,v] = [-\infty, +\infty] = \mathbb{R}$. If $\mathbf{p}_n \neq \mathbf{0}$, then the **degree** of $\mathbf{f}$ is n. Alternatively, we can view $\mathbf{f}$ in terms of its coordinate functions, $\mathbf{f}(t) = (f_1(t), \ldots, f_d(t))$ where each $f_i(t)$ is a polynomial of degree n or less.

We begin with a particular representation of polynomial parametric curves, called Bézier curves. To obtain our first intuition, let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be three distinct points. Consider the points parametrized by a real value t,

$$\mathbf{a}^1(t) := (1-t)\mathbf{a} + t\mathbf{b}, \qquad \mathbf{b}^1(t) := (1-t)\mathbf{b} + t\mathbf{c}$$

and also

$$\begin{aligned} \mathbf{a}^2(t) &= (1-t)\mathbf{a}^1(t) + t\mathbf{b}^1(t) \\ &= (1-t)^2\mathbf{a} + 2t(1-t)\mathbf{b} + t^2\mathbf{c}. \end{aligned}$$

We call $\mathbf{a}^2(t)$ the "Bézier curve generated by the control points $\mathbf{a}, \mathbf{b}, \mathbf{c}$". It is easy to see that $\mathbf{a}^2(0) = \mathbf{a}$ and $\mathbf{a}^2(1) = \mathbf{c}$, and the curve lies in the triangle $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ when t is restricted to $[0,1]$. To see that this curve is a parabola, first consider the case where $\mathbf{a} = (-1,1), \mathbf{b} = (0,-1)$ and $\mathbf{c} = (1,1)$. Let $\mathbf{a}^2(t) = (x(t), y(t))$. Then we have

$$x(t) = (1-t)^2(-1) + t^2(1) = 2t - 1, \qquad y(t) = (2t-1)^2$$

Thus we have the parabola $y = x^2$. As noted below, Bézier curves are preserved by affine transformation. Hence the Bézier curve defined by any three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can be affinely transformed to the Bézier curve defined by $\mathbf{a} = (-1,1), \mathbf{b} = (0,-1), \mathbf{c} = (1,1)$. Since parabolas are preseved by affine transformations, this proves our claim.

The above process for generating a parabola from three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can be generalized. This idea was introduced by Paul de Casteljau (1959), and this process is called Casteljau's Algorithm. The algorithmic idea of Casteljau's can be said to be more important than the corresponding Bézier curves: the algorithmic idea can be fruitfully used in other contexts. Moreover, it is the existence of this algorithm that distinguishes Bézier curves as a useful tool in geometric modeling. See Farin [2] for an account of the discovery of these curves in the French automobile industry (Casteljau was at Citroen and Bézier at Renault).

Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^3$ be $n+1$ distinct points and let $t \in \mathbb{R}$. Define a doubly-indexed family of curves

$$\{\mathbf{p}_i^r(t) : r = 1, \ldots, n, i = 0, 1, \ldots, n-r\}$$

where

$$\mathbf{p}_i^r(t) = (1-t)\mathbf{p}_i^{r-1}(t) + t\mathbf{p}_{i+1}^{r-1}(t) \tag{1}$$

where $\mathbf{p}_i^0(t) = \mathbf{p}_i$ for all i and t. The superscript r indicates the "rank" of the curve, with the input points having rank 0. It turns out, the curves of rank r is a polynomial parametric curve of degree r. This is easily seen from (1) where the rank r curves are generated from rank $r-1$ curves. There are $n-r+1$ rank r curves.

Thus there is only one curve of rank n, namely, $\mathbf{p}_0^n(t)$. This is the **Bézier curve** defined (or generated) by $\mathbf{p}_0, \ldots, \mathbf{p}_n$. Call $\mathbf{p}_0, \ldots, \mathbf{p}_n$ the **knots** or **control points** for the curve. If $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$, then by a slight abuse of notation, we will write $\mathbf{p}(t)$ to denote the Bézier curved defined by $\mathbf{p}$. We may write $\mathbf{p}^r(t)$ instead of $\mathbf{p}_0^r(t)$ (dropping the 0 subscripts). Then $\mathbf{p}(t)$ is just $\mathbf{p}^n(t)$.

The family $\{\mathbf{p}_i^r(t)\}_{r,i}$ may be arranged in a triangular shape, with the rank r functions in column $r+1$. Illustrating this for $n = 3$:

$$
\begin{array}{cccc}
\mathbf{p}_0 & & & \\
 & \mathbf{p}_0^1 & & \\
\mathbf{p}_1 & & \mathbf{p}_0^2 & \\
 & \mathbf{p}_1^1 & & \mathbf{p}_0^3 \\
\mathbf{p}_2 & & \mathbf{p}_1^2 & \\
 & \mathbf{p}_2^1 & & \\
\mathbf{p}_3 & & &
\end{array}
\tag{2}
$$

Let us consider the rank r of curves. Rank 0 curves are constant functions, and rank 1 curves are linear functions. For instance, $\mathbf{p}_i^1(t)$ is the parametrized line passing through the points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$. As shown above, a rank 2 Bézier curve is a parabola. As expected from this progression, the rank r curves have algebraic degree r. This is seen from the recursive equation (1): if the curves b_i^{r-1} and b_{i+1}^{r-1} has degree $\leq r-1$, then b_i^r has degree $\leq r$.

We have the following **convex hull** property: for $0 \leq t \leq 1$, the point $\mathbf{p}(t)$ is contained in the convex hull of $\mathbf{p}_0, \ldots, \mathbf{p}_n$. More generally, we claim: *the point $\mathbf{p}_i^r(t)$ ($r \geq 1, i \geq 0$) is contained in the convex hull of $\mathbf{p}_i(t), \mathbf{p}_{i+1}(t), \ldots, \mathbf{p}_{i+r}(t)$.* This is true for $r = 1$. To see this for $r > 1$, note that the point $\mathbf{p}_i^{r+1}(t)$ is contained in the convex hull of $\mathbf{p}_i^r(t)$ and $\mathbf{p}_{i+1}^r(t)$. These two points, by induction, are contained in the convex hull of

$$\mathbf{p}_i(t), \mathbf{p}_{i+1}(t), \ldots, \mathbf{p}_{i+r}(t), \mathbf{p}_{i+r+1}(t).$$

Thus our claim follows.

Horner Scheme for Casteljau's Algorithm. Suppose we want to compute the point $\mathbf{p}(t)$ for a numerical value of t. We can give a Horner-like scheme for its evaluation as follows:

§3. Affine Maps and Convex Combinations

A function $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is called **affine** if it has the form

$$\alpha(\mathbf{x}) = \mathbf{b} + \mathbf{A}\mathbf{x}$$

for some vector $\mathbf{b}$ and $d \times d$ matrix $\mathbf{A}$. Such functions are completely defined by its actions on any $d+1$ points that do not lie on a hyperplane. The canonical set of points is $\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\}$ where $\mathbf{0}$ is the origin, and $\mathbf{e}_i$ is the elementary vector with 1 at the ith position and 0 elsewhere. Indeed, it is easily seen that $\mathbf{b} = \alpha(\mathbf{0})$ and the ith column of $\mathbf{A}$ is simply $\alpha(\mathbf{e}_i) - \alpha(\mathbf{0})$. If $\mathbf{A}$ is invertible, then the affine map is invertible.

Let $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ be points in $\mathbb{R}^d$. An expression of the form

$$\sum_{i=0}^{n} c_i \mathbf{p}_i$$

is called a **linear combination** of the points $\mathbf{p}_0, \ldots, \mathbf{p}_n$. If the coefficients c_i are constrained to satisfy $\sum_{i=0}^n c_i = 1$, then this is called a **affine combination**. Finally, the c_i's are non-negative in an affine combination, we call it a **convex combination**. The set of all linear (resp., affine) combinations of $\mathbf{p}$ forms a linear (resp., affine) subspace of $\mathbb{R}^d$. The set of convex combinations of $\mathbf{p}$ is a convex set; its boundary is called the **convex hull** of $\mathbf{p}$. A simple property of affine maps $\alpha(\mathbf{x}) = \mathbf{b} + \mathbf{A}\mathbf{x}$ is that it preserves affine combinations:

$$\alpha(\sum_{i=0}^{n} c_i \mathbf{p}_i) = \sum_{i=0}^{n} c_i \alpha(\mathbf{p}_i), \qquad \sum_{i=0}^{n} c_i = 1. \tag{3}$$

In proof:

$$\begin{aligned}\alpha(\sum_{i=0}^{n} c_i \mathbf{p}_i) &= \sum_{i=0}^{n} c_i(\mathbf{A}\mathbf{p}_i) + \mathbf{b}\\ &= \sum_{i=0}^{n} c_i(\mathbf{A}\mathbf{p}_i + \mathbf{b})\\ &= \sum_{i=0}^{n} c_i \alpha(\mathbf{p}_i)\end{aligned}$$

Let $\mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_n)$ where $\mathbf{q}_i = \alpha(\mathbf{p}_i)$ for all i. How are the Bézier curves $\mathbf{q}(t)$ and $\mathbf{p}(t)$ related? We have:

$$\alpha(\mathbf{p}_i^r(t)) = \mathbf{q}_i^t(t)$$

for all t, r, i. In short, Bézier curves are preserved by affine maps. The proof is immediate from (3), and the fact that Casteljau's algorithm uses affine combinations to construct rank r curves from rank $r-1$ curves.

This is related to the fact that affine maps preserves ratios:

$$Ratio(a, b, c) = Ratio(\alpha(a), \alpha(b), \alpha(c)).$$

§4. Bernstein Polynomials

Bézier curves are intimately connected to the Bernstein polynomials: for each natural number n, define

$$B_i^n(t) := \begin{cases} \binom{n}{i} t^i (1-t)^{n-i} & \text{for } \ i = 0, 1, \ldots, n\\ 0 & \text{for } \ i > n \text{ or } \ i < 0.\end{cases} \tag{4}$$

Also, $B_0^n(t) = (1-t)^n$ and $B_n^n(t) = t^n$. In particular, when $n = 0$, $B_0^0(t)$ is the constant 1. For $n = 1$, $B_0^1(t) = (1-t)$ and $B_1^1(t) = t$. For $n = 2$,

$$B_0^2(t) = (1-t)^2, \quad B_1^2(t) = 2(1-t)t, \quad B_2^2(t) = t^2.$$

It is clear from the definition that $B_i^n(t)$ is a polynomial of degree n.

To see connection to the Bézier curves, consider the second degree case:

$$\begin{aligned}\mathbf{p}_0^2(t) &= (1-t)\mathbf{p}_0^1(t) + t\mathbf{p}_1^1(t)\\ &= (1-t)[(1-t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1-t)\mathbf{p}_1 + t\mathbf{p}_2]\\ &= (1-t)^2\mathbf{p}_0 + 2(1-t)t\mathbf{p}_1 + t^2\mathbf{p}_2\\ &= B_0^2(t)\mathbf{p}_0 + B_1^2(t)\mathbf{p}_1 + B_2^2(t)\mathbf{p}_2.\end{aligned}$$

In general, the Bézier curve $\mathbf{p}(t)$ generated by $\mathbf{p}_0, \ldots, \mathbf{p}_n$ is given by

$$\mathbf{p}(t) = \sum_{j=0}^{n} B_j^n(t)\mathbf{p}_j. \tag{5}$$

More generally, we have

$$\mathbf{p}_i^r(t) = \sum_{j=0}^{r} B_j^r(t)\mathbf{p}_{i+j}. \tag{6}$$

This is easily verified by induction. The expression (5) is called the **Bernstein form** of the Bézier curve $\mathbf{p}(t)$. From (5), we see that a Bézier curve of degree n is a weighted sum of the knot points $\mathbf{p}_0, \ldots, \mathbf{p}_n$. The weights are given by the n Bernstein polynomials. Conversely, we will show below that every polynomial parametric curve is a Bézier curve.

Let us consider the behavior of the polynomial $B_j^n(t)$ in the range $0 \le t \le 1$. As t increases from 0 to t, the function $B_0^n(t)$ decreases from 1 to 0, $B_n^n(t)$ increases from 0 to 1. But in general, $B_j^n(t)$ is non-negative ($B_j^n(t) \ge 0$) and a unimodal function which attains its maximum when $t = i/n$. This is obvious for $i = 0$ or $i = n$. For $0 < i < n$, this is shown next, with other properties of the Bernstein polynomials.

Symmetry We have: $B_j^n(t) = B_{n-j}^n(1-t)$. This comes directly from the definition.

Partition of unity This means $\sum_{j=0}^{n} B_j^n(t) = 1$. Writing $\overline{t}$ for $1-t$, we have

$$1 = (t + \overline{t})^n = \sum_{j=0}^{n} \binom{n}{j} t^j \overline{t}^{n-j} = \sum_{j=0}^{n} B_j^n(t).$$

Combined with the fact that $B_j^n(t) \ge 0$, we conclude from (1) that $\mathbf{p}(t)$ is a convex combination of the points $\mathbf{p}_0, \ldots, \mathbf{p}_n$. This gives another proof of the convex hull property.

Recurrence The degree n polynomials can be expressed as an affine combination of the degree $n-1$ polynomials: $B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$. This follows from the standard identity,

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}.$$

Thus,

$$\begin{aligned}
B_i^n(t) &= \binom{n}{i} t^i (1-t)^{n-i} \\
&= \binom{n-1}{i} t^i (1-t)^{n-i} + \binom{n-1}{i-1} t^i (1-t)^{n-i} \\
&= (1-t)\binom{n-1}{i} t^i (1-t)^{n-1-i} + t\binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} \\
&= (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t).
\end{aligned}$$

Derivative We have:

$$\frac{d}{dt} B_i^n(t) = n\left[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right]. \tag{7}$$

The derivation is rather similar to the previous one. If we equate (8) to 0 and solve for t, we obtain $t = i/n$. This point cannot be a minima (for $0 < i < n$, we use the fact that $B_i^n(t) \ge 0$ with equality at $t = 0, 1$). This proves our earlier assertion that $B_i^n(t)$ attains its maximum at $t = i/n$.

Applying this to the derivative of the Bézier curve $\mathbf{p}(t)$ in (5), we obtain

$$\frac{d}{dt}\mathbf{p}(t) = n\sum_{j=0}^{n-1} (\mathbf{p}_{j+1} - \mathbf{p}_j) B_j^{n-1}(t), \tag{8}$$

which is another Bézier curve.

Polynomial Basis: Note that the set $\{1, t, t^2, \ldots, t^n\}$ of monomials is a basis for the vector space of polynomials of degree at most n. We now prove that $\{B_i^n : i = 0, \ldots, n\}$ is also a basis. This from the following

$$\binom{n}{i} t^i = \sum_{j=i}^{n} \binom{j}{i} B_j^n(t) \tag{9}$$

In proof, let us expand both sides using factorials:

$$\begin{array}{lcl}
\frac{n!}{i!(n-i)!} t^i & = & \sum_{j=i}^{n} \frac{j!}{i!(j-i)!} B_j^n(t) \\
 & = & \sum_{j=i}^{n} \frac{j!}{i!(j-i)!} \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j} \\
t^i & = & \sum_{j=i}^{n} \frac{(n-i)!}{(j-i)!(n-j)!} t^j (1-t)^{n-j} \\
1 & = & \sum_{j=i}^{n} \frac{(n-i)!}{(j-i)!(n-j)!} t^{j-i} (1-t)^{n-j} \\
 & = & \sum_{j=0}^{n-i} \frac{(n-i)!}{j!(n-i-j)!} t^j (1-t)^{n-i-j} \\
 & = & \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} t^j (1-t)^{m-j} \\
 & = & \sum_{j=0}^{m} B_j^m(t)
\end{array}$$

where $m = n - i$. But the final expression is 1 by partition of unity.

Subdivision This says

$$B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t).$$

Product We have

$$B_i^m(t) B_j^n(t) \binom{m+n}{i+j} = \binom{m}{i} \binom{n}{j} B_{i+j}^{m+n}(t).$$

Degree Elevation There are three formulas that expresses $B_i^n(t)$ in terms of degree $n+1$ Bernstein polynomials.

§5. Spline Curves

A spline curve

$$s : [u_0, u_n] \to \mathbb{R}^3$$

is a continuous curve such that there exists u_i, $u_0 < u_1 < \cdots < u_{n-1} < u_n$, (called breakpoints) such that $s[u_i, u_{i+1}]$ is a polynomial curve. We consider the case where the $s[u_i, u_{i+1}]$ are Bézier curves.

Subdivision. Let $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ are given and $0 < u < 1$ is given. Consider the subdivision problem: find $\mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_n)$ such that $\mathbf{q}[0, 1] = \mathbf{p}[0, u]$. The **subdivision property** for Bézier curves says:

$$\mathbf{q}_i = \mathbf{p}_0^i(u), \qquad i = 0, \ldots, n$$

For instance, $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{q}_1 = \mathbf{p}_0^1(u) = (1-u)\mathbf{p}_0 + t\mathbf{p}_1$. This is illustrated in figure 2.

Extrapolation. The converse of the subdivision problem is the following: Given $\mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_n)$ and $0 < u < 1$, we want to find $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ such that $\mathbf{q}[0, 1] = \mathbf{p}[0, u]$.

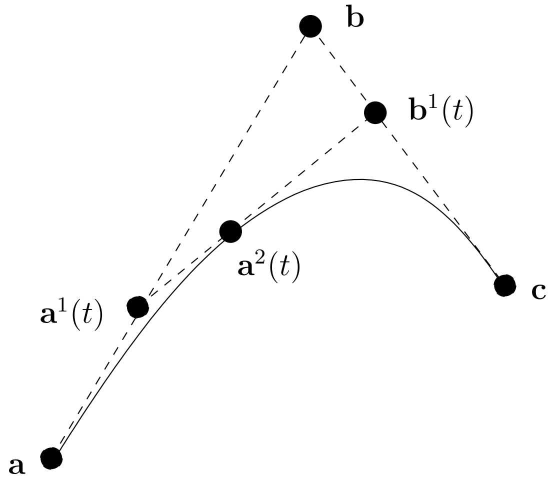


Figure 2: Casteljau's algorithm to plot the point $\mathbf{a}^2(t)$.

Smoothness. Suppose $\mathbf{s}$ is a spline curve that the concatenation of the Bézier curves $\mathbf{s}_1 = \mathbf{s}[u_0, u_1]$ and $\mathbf{s}_2 = \mathbf{s}[u_1, u_2]$ defined (respectively) by $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ and $\mathbf{p}' = (\mathbf{p}_n, \ldots, \mathbf{p}_{2n})$. Then we have the theorem:

The curve $\mathbf{s}$ is C^r at $\mathbf{p}_n$ if and only if $\mathbf{p}_{n+i} = b^i_{n-i}(t)$ for $i = 0, \ldots, r$ and $t = (u_2 - u_0)/(u_1 - u_0)$.

§6. Spline Surfaces

Let $\mathbf{P} = (\mathbf{p}_{0,0}, \mathbf{p}_{0,1}, \mathbf{p}_{1,0}, \mathbf{p}_{1,1}) \in \mathbb{R}^3$. The surface

$$x(u, v) = \sum_{i=0}^{1} \sum_{j=0}^{1} \mathbf{p}_{i,j} B^1_i(u) B^1_j(v)$$

is the **hyperbolic paraboloic** defined by $\mathbf{P}$. We now need to use the matrix notation to achieve a compact notation: thus we write

$$x(u, v) = [(1-u), u] \cdot \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} \end{bmatrix} \cdot \begin{bmatrix} (1-v) \\ v \end{bmatrix}.$$

More generally, if

$$\mathbf{P} = (\mathbf{p}_{i,j} : i, j = 0, \ldots, n)$$

and $(u, v) \in \mathbb{R}^2$, we have

$$\mathbf{p}^r_{i,j}(u, v) = [(1-u), u] \cdot \begin{bmatrix} \mathbf{p}^{r-1}_{i,j} & \mathbf{p}^{r-1}_{i,j+1} \\ \mathbf{p}^{r-1}_{i+1,j} & \mathbf{p}^{r-1}_{i+1,j+1} \end{bmatrix} \cdot \begin{bmatrix} (1-v) \\ v \end{bmatrix}, \qquad (r = 1, \ldots, n, \quad i, j = 0, \ldots, n-r).$$

We call $\mathbf{P}$ above a **control net**. The above $\mathbf{P}$ has a square shape, which could be generalized to a rectangular shape.

Bézier Triangles. We can also use triangular control nets. To do this, we need to use barycentric coordinates. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the vertices of a triangle in the plane, then for any point $\mathbf{p} \in \mathbb{R}^2$ can be uniquely expressed as $\mathbf{p} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}$ where (u, v, w) are three real numbers such that $u + v + w = 1$. We call $\mathbf{u} = (u, v, w)$

the **barycentric coordinates** of $\mathbf{p}$ (with respect to $\mathbf{a}, \mathbf{b}, \mathbf{c}$). Furthermore, $\mathbf{p}$ is inside the triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ if and only if u, v, w are all non-negative. Let $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be the signed area of the triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ given by half the determinant of the matrix $(\mathbf{a}_1|\mathbf{b}_1|\mathbf{c}_1)$ whose the columns are $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1$ where in general, $\mathbf{a}_1$ is obtained by appending a 1 to the end of the vector $\mathbf{a}$. Then we have

$$u = \Delta(\mathbf{p}, \mathbf{b}, \mathbf{c})/\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}), v = \Delta(\mathbf{a}, \mathbf{p}, \mathbf{c})/\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}), w = \Delta(\mathbf{a}, \mathbf{b}, \mathbf{p})/\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

Let

$$\mathbf{P} = (\mathbf{p}_{i,j,k} : i + j + k = n; i, j, k \geq 0\}$$

This control net has $(n + 1)(n + 2)/2$ control points. To illustrate this, consider the case $n = 4$ which has 15 control points:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{p}_{040} & & & & \\
 & & & \mathbf{p}_{031} & & \mathbf{p}_{130} & & & \\
 & & \mathbf{p}_{022} & & \mathbf{p}_{121} & & \mathbf{p}_{220} & & \\
 & \mathbf{p}_{013} & & \mathbf{p}_{112} & & \mathbf{p}_{211} & & \mathbf{p}_{310} & \\
\mathbf{p}_{004} & & \mathbf{p}_{103} & & \mathbf{p}_{202} & & \mathbf{p}_{301} & & \mathbf{p}_{401}
\end{array} \qquad (10)$$

In general, let $\mathbf{i} = (i, j, k)$ denote an index triple, and write $|\mathbf{i}| = i + j + k$. Also, let $\mathbf{e}_i$ be the triple with 1 at the ith position and zero elsewhere. If $\mathbf{u} = (u, v, w)$ is a barycentric coordinate, we define the functions

$$\mathbf{p}_{\mathbf{i}}^{r}(\mathbf{u}) = u\mathbf{p}_{\mathbf{i}+\mathbf{e}_1}^{r-1}(\mathbf{u}) + v\mathbf{p}_{\mathbf{i}+\mathbf{e}_2}^{r-1}(\mathbf{u}) + w\mathbf{p}_{\mathbf{i}+\mathbf{e}_3}^{r-1}(\mathbf{u}), \qquad |\mathbf{i}| = n - r; r = 1, \ldots, n.$$

The surface $\mathbf{P}(u, v, w) = \mathbf{p}_0^n i(u, v, w)$ is called the **Bézier triangle** generated by the control net $\mathbf{P}$.

Exercises

Exercise 6.1: Show that if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are distinct points on a parabola and the tangents at these points meet at $t(\mathbf{p}, \mathbf{q}), t(\mathbf{q}, \mathbf{r})$ and $t(\mathbf{p}, \mathbf{r})$,

respectively, then *ratio*() ... ♢

Exercise 6.2: (Linear Precision)

$$\sum_{j=0}^{n} \frac{j}{n} B_j^n(t) = t.$$

Give an interpretation of this formula in terms of Bézier curves: consider the control points $\mathbf{p}_0, \ldots, \mathbf{p}_n$ where are uniformly distributed along the straightline segment connecting $\mathbf{p}_0$ and $\mathbf{p}_n$. ♢

Exercise 6.3: Program the Horner-like schme for evaluating points on Bézier curves (use your favorite programming language). ♢

End Exercises

References

[1] V. G. Boltyanskiĭ and V. A. Efremovich. *Intuitive Combinatorial Topology*. Springer, 2001. Translated by Abe Shenitzer.

[2] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Academic Press, Inc, second edition, 1990.