" ...self-amortizing canals... "

– Mr. Banks in *Mary Poppins* (the movie)

Lecture VI
AMORTIZATION

Amortization is the idea of distributing cost over a period of time, as when we take a 25-year bank mortgage loan[1] to pay for a home. Similar ideas can be used to analyze the cost when we run an algorithm. Suppose each run of the algorithm amounts to a sequence of operations on a data structure. For instance, to sort n items, the well-known **heapsort algorithm** (Lect.III,¶7) is a sequence of n `insert`'s into an initially empty priority queue, followed by a sequence of n `deleteMin`'s from the queue until it is empty. Thus if c_i is the cost of the ith operation, the algorithm's running time is $\sum_{i=1}^{2n} c_i$, since there are $2n$ priority queue operations in all. In worst case analysis, we ensure that *each* operation is efficient, say $c_i = O(\log n)$, leading to the conclusion that the overall algorithm is $O(n \log n)$. The idea of **amortization** exploits the fact that we may be able to obtain the same bound $\sum_{i=1}^{2n} c_i = O(n \log n)$ *without* ensuring that each c_i is logarithmic. We then say that the **amortized cost** of each operation is logarithmic. Thus "amortized complexity" is a kind of average complexity although it has nothing to do with probability. Tarjan [11] gave the first systematic account of this topic.

¶1. Why amortize? Note that for the heapsort problem above, we could have ensured that *each* operation is logarithmic time. Nevertheless, it may be advantageous to consider algorithms that only achieve logarithmic behavior only in the amortized sense. For instance, the extra flexibility of using amortized bounds may lead to simpler or more practical algorithms. Indeed many "amortized" data structures are relatively easy to implement. To give a concrete example, consider any balance binary search tree scheme. The algorithms for such trees must perform considerable book-keeping to maintain its balanced shape. In contrast, the splay trees in this chapter provide an amortization scheme for binary search trees that is considerably simpler and "lax" about balancing. The operative word in amortization is[2] laziness: try to defer the book-keeping work to the future, when it might be more convenient to do this work.

This chapter is in 3 parts: we begin by introducing the **potential function framework** for doing amortization analysis. Then we introduce two data structures, **splay trees** and **Fibonacci heaps**, which can be analyzed using this framework. We give a non-trivial application of each data structure: splay trees are used to maintain the convex hull of a set of points in the plane, and Fibonacci heaps are used to implement Prim's algorithm for minimum spanning trees.

§1. The Potential Framework

[1] Your home is said to be mortgaged (serves as collateral for the loan). Actually, this mortgaging analogy fails in the details. As we will see, our amortization model is akin to a banking model where you maintain an balance account that must not go negative.

[2] In algorithmics, it appears that we like to turn conventional vices (greediness, laziness, gambling with chance, etc) into virtues.

We formulate an approach to amortized analysis using the concept of "potential functions". Borrowing a concept from Physics, we imagine data structures as storing "potential energy" that can be released to do useful work. First, we view a data structure such as a binary search tree as a persistent object that has a state which can be changed by operations (e.g., insert, delete, etc). The *characteristic property* of potential functions is that they are a function of the *current* state of the data structure alone, independent of the history of how the data structure was derived. For instance, consider a binary search tree T_1 obtained by inserting the keys $1, 2, 3$ (in some order) into an empty tree and a tree T_2 obtained by inserting the keys $1, 2, 3, 4$, followed by deletion of 4. If T_1 and T_2 have the same shape, then the potential energy in these two trees will be the same.

Counter example to what?

¶2. A "Counter Example". Let C be a binary counter and the only operation on C is to increment its value. Starting with a counter value of 0, our counter C goes through the following sequence of states as we repeatedly increment it:

$$(0) \to (01) \to (010) \to (011) \to (0100) \to (0101) \to \ldots$$

Note that we use the convention of prepending a 0 bit to the standard binary notation for an integer.

The problem we consider is to bound the cost of a sequence of n increments, starting from an initial counter value of 0. In the worst case, an increment operation costs $\Theta(\lg n)$. Therefore a worst-case analysis would conclude that the total cost is $O(n \lg n)$. We can do better by using amortized analysis.

We may assume C is represented by a sufficiently long binary array, or[3] alternatively, a linked list of 0 and 1's. This representation determines our **cost model**: the cost of an increment operation is the number of bits we need to flip. Note the the number of bits to flip is the length of the suffix of C of the form 01^*, i.e., 0 followed by zero or more occurrences of 1. Since C always begins with a 0-bit, a suffix of the form 01^* always exists. This cost is at least 1. For instance, if the value of C is 27, then $C = (011011)$. After incrementing, we get $C = (011100)$. The cost of this increment is 3 since C has the suffix 011 before the increment. Note that we may also need to prepend a 0 to C when our increment operation flip the leftmost 0 to a 1: E.g., if $C = (0111)$ and after incrementing, we get $C = (1000)$. Now we need to prepend a 0 bit to the counter to get $C = (01000)$. The prepending cost is $O(1)$ and will be absorbed into the overall cost to flip bits; so we still say the cost for incrementing $C = (0111)$ is 4.

To begin our amortized analysis, we associate with C a **potential** $\Phi = \Phi(C)$ that is equal to the number of 1's in its list representation. In our preceding example, $\Phi(011011) = 4$ and $\Phi(011100) = 3$; so the potential of C decreased by 1 after this particular increment. Informally, we will "store" $\Phi(C)$ units of work (or energy) in C. To analyze the increment operation, we consider two cases.

- (i) Suppose the least significant bit of C is 0. Then the increment operation just changes this bit to 1. Note that the potential increases by 1 by this operation. We can **charge** this operation 2 units – one unit to do the work and one unit to pay for the increase in potential.
- (ii) Suppose an increment operation changes a suffix $\underbrace{0111\cdots11}_{k}$ of length $k \geq 2$ into $\underbrace{1000\cdots00}_{k}$: the cost incurred is k. Notice that the potential Φ decreases by $k-2$. This decrease "releases" $k-2$ units of work that can pay for $k-2$ units of the incurred cost. Hence we only need to charge this operation 2 units to make up the difference.

Thus, in both cases (i) and (ii), we charge 2 units of work for an operation, and so the total charges over n operations is only $2n$. We conclude that the amortized cost is $O(1)$ per increment operation.

[3] The linked list interpretation becomes necessary if we want to do other operations with multiple counters efficiently. See Exercise 1.3.

¶3. Abstract Formulation. We now formulate an amortization analysis that captures the essence of the above Counter Example. It is assumed that we are analyzing the cost of a sequence

$$p_1, p_2, \ldots, p_n$$

of **requests** on a data structure D. We view a data structure D as comprising two parts: its **contents** and its **structure**, where the structure represents some organization of the contents. The (contents, structure) pair is called the **state** of the data structure. The term "request" is meant to cover two types of operations: **updates** that modify the contents of D and **queries** that computes a function of the contents. Queries do not modify the contents, and there is no logical necessity to modify the structure either. However, it may be advantageous to modify the structure during queries.

For example, D may be a binary search tree storing a set of keys; the contents of D are these keys and the structure of D is the shape of binary tree itself. Inserting into D is an update request, and looking up a key in D is a query request. In Lookups, it is clear that the D need not change. Nevertheless, to ensure a favorable complexity over a sequence of such operations, we will perform some rotations to bring the searched-for node nearer to the root. This is called the the "move-to-front" heuristic.

The data structure D is dynamically changing: each request transforms the current state of D. Let D_i be the state of the data structure *after* request p_i, with D_0 the initial state.

Each p_i has a non-negative **cost**, denoted $\textsc{Cost}(p_i)$. This cost depends on the complexity model — which is part of the problem specification. To carry out an amortization argument, we must specify a **charging scheme** and a **potential function**. Unlike the complexity model, the charging scheme and potential function are not part of the problem specification. They are artifacts of our analysis and may require their invention may require some ingenuity.

A **charging scheme** is just any systematic way to associate a real number $\textsc{Charge}(p_i)$ to each operation p_i. E.g., for our "Counter Example", we define $\textsc{Charge}(p_i) := 2$ for each p_i. Informally, we "levy" a **charge** of $\textsc{Charge}(p_i)$ on the operation. We emphasize that this levy need not have any obvious relationship to $\textsc{Cost}(p_i)$. The **credit** of this operation is defined to be the "excess charge",

$$\textsc{Credit}(p_i) := \textsc{Charge}(p_i) - \textsc{Cost}(p_i). \tag{1}$$

In view of this equation, specifying a charging scheme is equivalent to specifying a credit scheme. The credit of an operation can be a negative number (in which case it is really a "debit").

A **potential function** is a non-negative real function Φ on the set of possible states of D satisfying

$$\Phi(D_0) = 0.$$

We call $\Phi(D_i)$ the **potential** of state D_i. Let the *increase in potential* of the ith step be denoted by

$$\Delta\Phi_i := \Phi(D_i) - \Phi(D_{i-1}).$$

The amortization analysis amounts to verifying the following inequality at every step:

$$\textsc{Credit}(p_i) \geq \Delta\Phi_i. \tag{2}$$

We call (2) the **credit-potential invariant**.

OK, we are mixing financial and physical metaphors here. Since mortgages is a financial term, perhaps the credit/debit ought to be put into a "bank account" and Φ is the "current balance".

The idea is that credit is stored as "potential" in the data structure. Since the potential function and the charging scheme are defined independently of each other, the truth of the invariant (2) is not a foregone conclusion. It must be verified for each case.

If the credit-potential invariant is verified, we can call the charge for an operation its **amortized cost**. This is justified by the following derivation:

$$\begin{array}{rcll}
\sum_{i=1}^n \textsc{Cost}(p_i) & = & \sum_{i=1}^n (\textsc{Charge}(p_i) - \textsc{Credit}(p_i)) & \text{(by the definition of credit)} \\
& \leq & \sum_{i=1}^n \textsc{Charge}(p_i) - \sum_{i=1}^n \Delta\Phi_i & \text{(by the credit-potential invariant)} \\
& = & \sum_{i=1}^n \textsc{Charge}(p_i) - (\Phi(D_n) - \Phi(D_0)) & \text{(telescopy)} \\
& \leq & \sum_{i=1}^n \textsc{Charge}(p_i) & (\text{since } \Phi(D_n) - \Phi(D_0) \geq 0).
\end{array}$$

When invariant (2) is a strict inequality, it means that some credit is discarded and the analysis is not tight in this case. For our "counter" example, the invariant is tight in every case! This means that our preceding derivation is an equality at each step until the very last step (when we assume $\Phi(D_n) - \Phi(D_0) \geq 0$). Thus we have the exact cost of incrementing a counter from 0 to n is **exactly** equal to

$$\sum_{i=1}^n c_i = 2n - \Phi(D_n)$$

where $\Phi(D_n)$ is the number of 1's in the binary representation of n. E.g., for our "Counter Example", the cost incurred to reach the counter value $C = (011001)$ is exactly $2(25) - 3 = 47$.

The distinction between "charge" and "amortized cost" should be clearly understood: the former is a definition and the latter is an assertion. A charge can only be called an amortized cost if the overall scheme satisfies the credit-potential invariant. *Get this!*

¶4. So what is Amortization? In the amortization framework, we are given a sequence of n requests on a data structure. We are also given a cost model (this may be implicit) which tells us the true cost c_i for the ith operation. We want to upper bound the total cost $\sum_{i=1}^n c_i$. In an amortization analysis, we hope to achieve a bound that is tighter than what can be achieved by replacing each c_i by the worst case cost. This requires the ability to take advantage of the fact that the cost of each type of request is variable, but its variability can somehow be smoothened out by sharing cost across different operations. This sharing of costs can be done is various ways. In the potential framework, we are required to invent a charging scheme and a potential function. After verifying that the credit-potential invariant holds for each operation, we may conclude that the charge *is* an amortized cost.

The potential function can be generalized in several ways: it need not be defined just for the data structure, but could be defined for any suitable abstract feature. Thus, we might have one potential function Φ_j for the jth feature ($j = 1, 2, \ldots$). The charge for an operation could be split up in several ways, and applied to each of the potential functions Φ_j.

We illustrate this by giving an alternative argument for the amortized cost of incrementing binary counters: let us set up a "charge account" at each bit position of the binary counter: let A_i be the account at the ith smallest position (the least significant position is $i = 0$, the next most significant position is $i = 1$, etc). Each unit of work changes the value of a particular bit of the counter; if the ith bit is changed, we charge the account A_i. Note that A_0 is n times. The account A_1 is charged $\leq n/2$ times, and in general, the account A_i is charged $\leq n/2^i$ times. Hence the overall charges is at most $\leq n(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \leq 2n$. Hence the amortize cost per increment is ≤ 2.

Note that this charging scheme is actually simpler than the potential method, since we charge each operation the *exact* cost of the operation! We will return to these ideas in a later chapter on the Union Find data structure.

EXERCISES

Exercise 1.1: Our model and analysis of counters can yield the exact cost to increment from any initial counter value to any final counter value. Show that the number of work units to increment a counter from 68 to 125 is (exactly!) 110. ♢

Exercise 1.2: Recall that our model of cost for incrementing a binary counter.
(a) What is the *exact* number of work units to count from 0 to 9?
(b) What is the *exact* number of work units to count from 0 to n?
(c) What is the *exact* number of work units to count from m to n ($m \leq n$)? ♢

Exercise 1.3: Give an exact amortized analysis for incrementing a 3-ary counter. ♢

Exercise 1.4: A simple example of amortized analysis is the cost of operating a special kind of push-down stack. Our stack S supports the following two operations: $S.\texttt{push}(K)$ simply add the key K to the top of the current stack. But $S.\texttt{pop}(K)$ will keep popping the stack as long at the current top of stack has a key smaller than K (the bottom of the stack is assume to have the key value ∞). The cost for push operation is 1 and the cost for popping $m \geq 0$ items is $m + 1$.
(a) Use our potential framework to give an amortized analysis for a sequence of such push/pop operations, starting from an initially empty stack.
(b) How tight is your analysis? E.g., can it give the *exact cost*, as in our Counter Example?

REMARK: Such a stack is used, for instance, in implementing Graham's algorithm for the convex hull of a set of planar points (see Section 4 on convex hull in this chapter). ♢

Exercise 1.5: Let us generalize the example of incrementing binary counters. Suppose we have a collection of binary counters, all initialized to 0. We want to perform a sequence of operations, each of the type

$$\texttt{inc}(C), \quad \texttt{double}(C), \quad \texttt{add}(C, C')$$

where C, C' are names of counters. The operation $\texttt{inc}(C)$ increments the counter C by 1; $\texttt{double}(C)$ doubles the counter C; finally, $\texttt{add}(C, C')$ adds the contents of C' to C while simultaneously set the counter C' to zero. Show that this problem has amortized constant cost per operation.

We must define the cost model. The length of a counter is the number of bits used to store its value. The cost to double a counter C is just 1 (you only need to prepend a single bit to C). The cost of $\texttt{add}(C, C')$ is the number of bits that the standard algorithm needs to look at (and possibly modify) when adding C and C'. E.g., if $C = 11,1001,1101$ and $C' = 110$, then $C + C' = 11,1010,0011$ and the cost is 9. This is because the algorithm only has to look at 6 bits of C and 3 bits of C'. Note that the 4 high-order bits of C are not looked at: think of them as simply being "linked" to the output. Here is where the linked list representation of counters is exploited. After this operation, C has the value $11,1010,0011$ and C' has the value 0.
HINT: The potential of a counter C should take into account the number of 1's as well as the bit-length of the counter.

You couldn't do this with arrays!

♢

Exercise 1.6: In the previous counter problem, we define a cost model for $\texttt{add}(C, C')$ that depends only on the bit patterns in C and C'. In particular, the cost of $\texttt{add}(C, C')$ and $\texttt{add}(C', C)$ are the same. How can you implement the addition algorithm so that the cost model is justified? HINT: recall that counters are linked lists, and you must describe your algorithm in terms of list manipulation. ♢

Exercise 1.7: Generalize the previous exercise by assuming that the counters need not be initially zero, but may contain powers of 2. ♢

Exercise 1.8: Joe Smart reasons that if we can increment counters for an amortized cost of $\mathcal{O}(1)$, we should be able to also support the operation of "decrementing a counter", in addition to those in the previous exercise. This should have an amortized cost of $\mathcal{O}(1)$, of course.
(a) Can you please give Joe a convincing argument as to why he is wrong?
(b) Joe's intuition about the symmetry of decrement and increment is correct if we change our complexity model. Vindicate Joe by a showing a model where we can increment, decrement, add and double in $O(1)$ per operation. HINT: we allow our counters to store negative numbers. We also need a more general representation of counters.
(c) In your solution to (b), let us add another operation, testing if a counter value is 0. What is the amortized complexity of this operation? ♢

Exercise 1.9: Suppose we want to generate a lexicographic listing of all n-permutations (See Chapter 5 on generating permutations). Give an amortized analysis of this process. ♢

END EXERCISES

§2. Splay Trees

The **splay tree data structure** of Sleator and Tarjan [10] is a practical approach to implementing all ADT operations listed in §III.2. Splay trees are just ordinary binary search trees – they are only distinguished by the algorithms used to implement standard binary tree operations. These operations invariably contain a splaying operation. The splaying operation, applied to an arbitrary node of the tree, will bring this node to the root position. We emphasize that the shape of a splay tree is arbitrary. For instance, it could be a tree consisting of a single path (effectively a list), as shown in the leftmost tree in Figure 1.

Splaying may be traced to an idea called the **move-to-front heuristic**: suppose we want to repeatedly access items in a list, and the cost of accessing the item is proportional to its distance from the front of the list. The heuristic says that it is a good idea to move an accessed item to the front of the list. Intuitively, this move will facilitate future accesses to this item. Of course, there is no guarantee that we would want to access this item again in the future. But even if we never again access this item, we have not lost much because the cost of moving the item has already been paid for (using the appropriate accounting method). Amortization (and probabilistic) analysis can be used to prove that this heuristic is a good idea. See Appendix A.

The analogue of the move-to-front heuristic in maintaining binary search trees is this: after accessing a key K in a tree T, we must move the node containing K to the root. This can be done by rotations, of course. What if we looked up K, and it is not in T? It is still essential to perform this heuristic (otherwise, it is clearly impossible to achieve a sublinear-time amortized bound). In this case, we move the successor or predecessor of K to move to the root. Recall that the **successor** of K in T is the smallest key K' in T such that $K \leq K'$; the **predecessor** of K in T is the largest key K' in T such that $K' \leq K$. Thus K does not have a successor (resp., predecessor) in T if K is larger (resp., smaller) than any key in T. Also, the successor and predecessor coincide with K iff K is in T. We characterize the **splay** operation as follows:

$$\texttt{splay}(\textit{Key } K, \textit{Tree } T) \to T' \tag{3}$$

re-structures the binary search tree T into an equivalent binary search tree T' in which the key K' at the root of T' is *either* the successor *or* predecessor of K in T. We are indifferent as to whether K' is the successor or predecessor. In particular, if K is smaller than any key in T, then K' is the smallest key in T. A similar remark applies if K is larger than any key in T. If T is non-empty, then any key K must have either a successor or predecessor in T, and thus the splay operation is well-defined on non-empty binary search trees. For example, starting from the BST in Figure 1(a), splaying the key $K = 4$ yields the BSF in Figure 1(c).

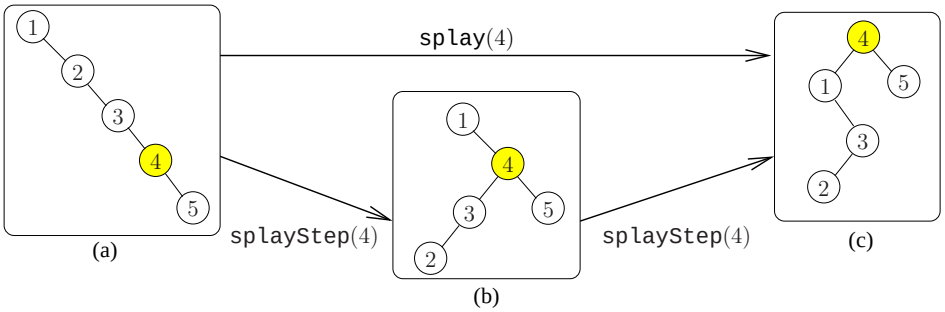


Figure 1: Splaying key 4 (with intermediate step)

Two variants of splaying: `splay`(K, T) *and* `splay`(u, T)

So far, we have described `splay`(K, T) where K is a key. We describe another variant where K is replaced by a node u in T: `splay`(u, T). The operation of `splay`(u, T) is to bring the node u to the root of T by repeated rotation; it returns the new tree T'. Then `splay`(K, T) is reduced to `splay`(u, T) as follows.

```
splay(K,T):
    u ← lookUp(K,T)
    Return splay(u,T)
```

Recall that `lookUp`(K, T) to find a node u in which u.`key` is the successor or predecessor of K in T. Note that K is in T iff u.`key` is equal to K. Before describing how `splay`(u, T) is implemented, we show how the splay operation is used.

¶5. Reduction of ADT operations to Splaying. We now implement the operations of the fully mergeable dictionary ADT (§III.2). The implementation is quite simple: each ADT operation is reducible to one or two splaying operations plus some easy $O(1)$ operations.

- `lookUp`(*Key* K, *Tree* T): first perform `splay`(K, T). Then examine the root of the resulting tree to see if K is at the root. `lookUp` is a success iff K is at the root. Note that we deliberately modify the tree T by splaying it.
- `insert`(*Item* X, *Tree* T): perform the standard binary search tree insertion of X into T. Note that this insertion would not be successful if X.`key` is already in T. In any case, the insertion process yields a node u whose key is equal to X.`key` (this key may either be newly inserted or was already there). Now splay this node u using our second variant of splaying, `splay`(u, T).

  Alternatively, we could first `splay`$(X$.`key`$, T)$, and then insert X in a straightforward way. See Exercises.

- `merge`(*Tree* T_1, T_2) $\to T$: recall that each key in T_1 must be less than any key in T_2. First let $T \leftarrow$ `splay`$(+\infty, T_1)$. Here $+\infty$ denotes an artificial key larger than any real key in T_1. So the root of T has no right child. We then make T_2 the right subtree of T.
- `delete`(*Key* K, *Tree* T): first perform `splay`(K, T). If the root of the resulting tree does not contain K, there is nothing to delete. Otherwise, delete the root and merge the left and right subtrees, as described in the previous bullet. This is illustrated in Figure 2.

  Alternatively, we can first do the standard deletion algorithm. This deletion identifies a node u whose child has been removed. We then do `splay`(u, T).

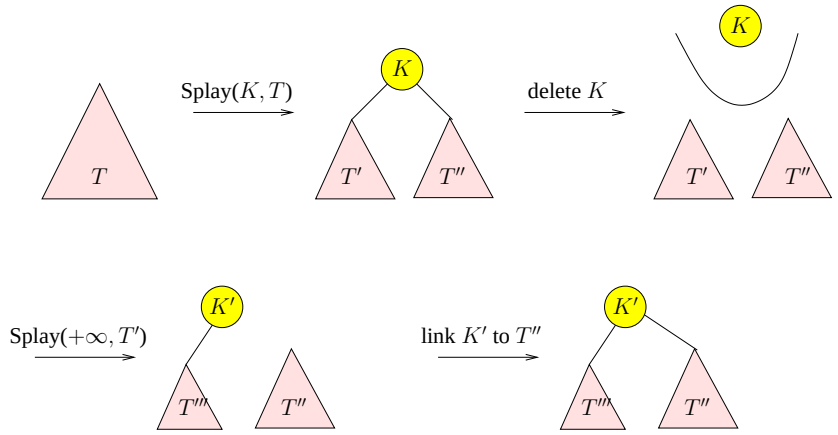


Figure 2: Steps in `delete`(K, T)

- `deleteMin`(*Tree* T): we perform $T' \leftarrow$ `splay`$(-\infty, T)$ and return the right subtree of T'.
- `split`(*Key* K, *Tree* T) $\to T'$: perform `splay`(K, T) so that the root of T now contains the successor or predecessor of K in T. Split off the right subtree of T, perhaps including the root of T, into a new tree T'.

It is interesting to compare the above ADT implementations with our treatment of Binary Search Trees in Chapter 3: there, we reduce all ADT operations to a single operation, rotation. This is still the case with splaying, except that we have hidden the rotations inside the splay operation.

¶6. Reduction to SplayStep. Let u be a node in a splay tree T. The operation `splay`(u, T) amounts to repeated application of a single operation

`splayStep`(u)

until u becomes the root of T. Termination is guaranteed if we insist that `splayStep`(u) always reduce the depth of any non-root u. There is an obvious interpretation of `splayStep`(u): it is simply "`rotate`(u)". Call this "naive splaying". Naive splaying will certainly validate the correctness of our splay algorithms in ¶5. The problem is that it defeats our amortization goal, which is to show that *each of the operations in ¶5 has an amortized cost of* $O(\log n)$.

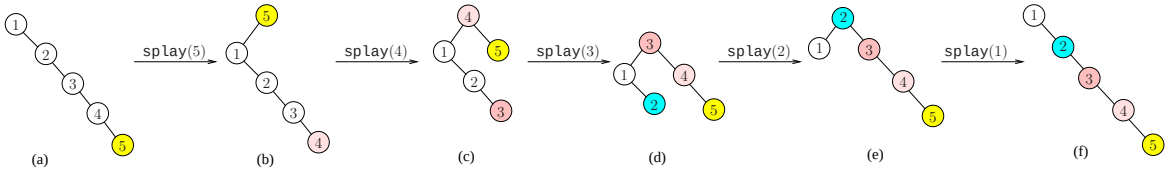


Figure 3: Naive splaying on T_5

The trouble with naive splaying. Consider the binary tree T_n that contains the keys $1, 2, \ldots, n$ and whose shape is just a right-path. The case T_5 is illustrated in Figure 3(a). It is easy to verify that we can get T_n by inserting the keys in the following order:

$$n, n-1, n-2, \ldots, 2, 1.$$

We remind the reader that this insertion follows the prescription of ¶5: first do standard insertion and then splay the inserted node.
Next, let us perform a sequence of Lookups on T_n: first do `lookUp`(n, T_n). Using our splay-based lookup algorithm, we first `splay`(n, T_n), which produces a tree rooted at n with a left child that is T_{n-1}. This is illustrated in Figure 3(b). We can repeat this process by performing lookups on $n-1$, $n-2, \ldots, 1$. As seen in Figure 3(f), the final result is again T_n. The cost for this sequence of operations is Θ-order of $\sum_{i=2}^{n} i = \Theta(n^2)$. Therefore it is impossible to achieve an amortized cost of $O(\log n)$ on each operation in this Lookup sequence.

The preceding example of naive splaying shows that the correct implementation of `splayStep` is quite subtle. We need a terminology: A grandchild u of a node v is called a **outer left grandchild** if u is the left child of the left child of v. Similarly for **outer right grandchild**. So an **outer grandchild** is either an outer left or outer right grandchild. If a node has a grandparent and is not an outer grandchild, then it is a **inner grandchild**.

`splayStep`(*Node* u):
There are three cases.
Base Case. If u.`parent` is the root,
then we simply `rotate`(u) (see Figure 9).
Case I. Else, if u is an outer grandchild,
perform two rotations: `rotate`$(u$.`parent`$)$, followed by `rotate`(u). See Figure 4.
Case II. Else, u is an inner grandchild and
we perform a double rotation (`rotate`(u) twice). See Figure 4.

In Figure 1, we see two applications of `splayStep`(4). Sleator and Tarjan calls the three cases of SplayStep the zig (base case), zig-zig (case I) and zig-zag (case II) cases. It is easy to see that the depth of u decreases by 1 in a zig, and decreases by 2 otherwise. Hence, if the depth of u is h, the splay operation will halt in $\lceil h/2 \rceil$ `splayStep`'s. Recall in §III.6, we call the zig-zag a "double rotation".

We illustrate the fact that `splay`(K, T) may return the successor or predecessor: let T_0 be the splay tree in Figure 5. If we call `splay`$(6, T_0)$, the result will be T_1 in the figure, where u.*Key* $= 7$. But if we call `splay`$(6, T_1)$, the result will be the tree T_2 in the figure, where u.*Key* $= 5$. What if you call `splay`$(6, T_2)$?

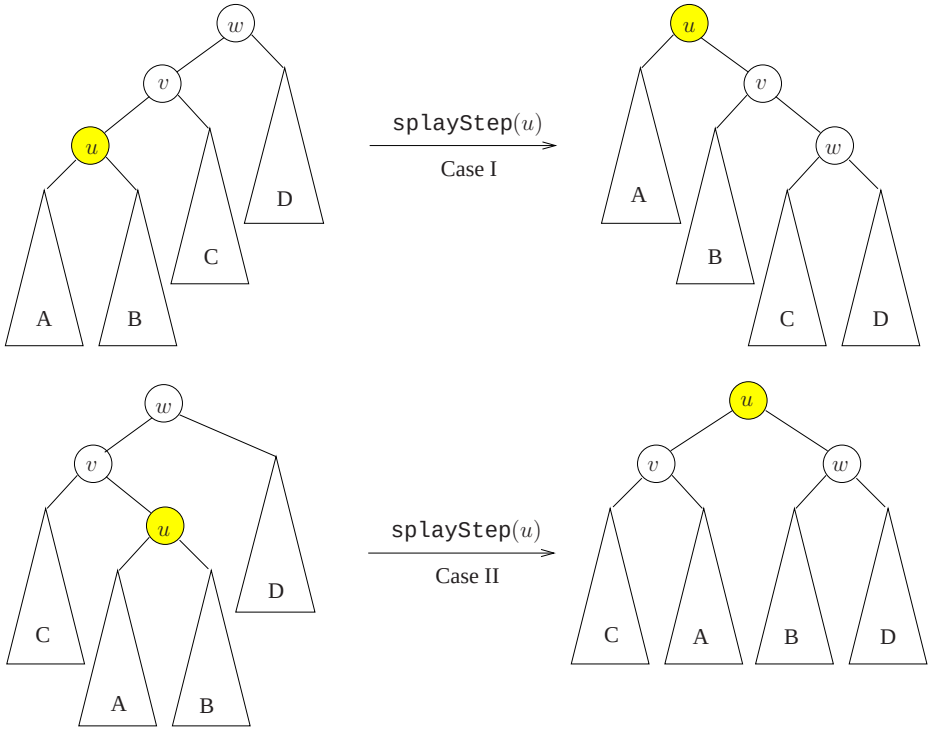


Figure 4: SplayStep at u: Cases I and II.

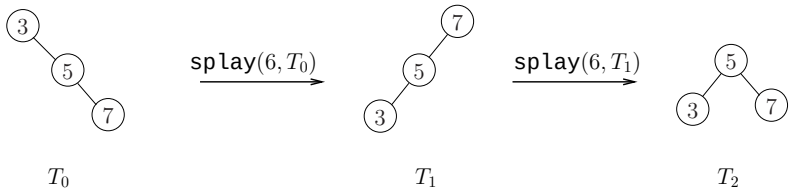


Figure 5: Splaying may return successor or predecessor

¶7. Top Down Splaying. We introduce a variation of splaying. The above splay algorithms require two passes over the splay path. Suppose we wish to have a one-pass algorithm, denoted `topSplay`(*Key* K, *Node* u).

The basic idea is to avoid the 2-phase approach of (1) going down the tree to look for K, and then (2) to bring the node which contains the successor or predecessor of K up to the root. Instead, we combine the 2 phases by bring each node that we wish to visit to the root *before* we "visit" it. This is called "top-down splaying".

Before giving the correct solution, it is instructive to illustrate some false starts. Suppose we are looking for the key K and u is the node at the root. If $u.Key = K$, we are done. If not, we must next visit the left or right child (u_L or u_R) of u. But instead of going down the tree, we simply rotate u_L or u_R to the root! For instance, if $u.Key > K$, then we must search for K in the subtree rooted at u_L, and so we do the rotation `rotate`(u_L) to bring u_L to the root. So our initial idea amounts to a repeated left- or right-rotation at the root. Unfortunately, this simple approach has a pitfall. To fix this, let us store some state information: we have 4 possible states in our algorithm. Again, u will be the root node and u_L, u_R denote the left and right child of u (these children may be null).

Find the pitfall!

- State 0: Both u_L and u_R have not been visited.
- State 1: u_L, but not u_R, has been visited. Moreover, u_L.`key` $< K$.
- State 2: u_R, but not u_L, has been visited. Moreover, u_R.`key` $> K$.
- State 3: Both u_L and u_R have been visited. Moreover, u_K.`key` $< K < u_R$.`key`.

We have a global variable `State` that is initialized to 0. Here are the possible state transitions. From state 0, we must enter either state 1 or state 2. From states 1 or 2, we can either remain in the same state or enter state 3. Once state 3 is entered, we remain in state 3. Unfortunately this `topSplay` algorithm does not have amortized logarithmic behavior (Exercise).

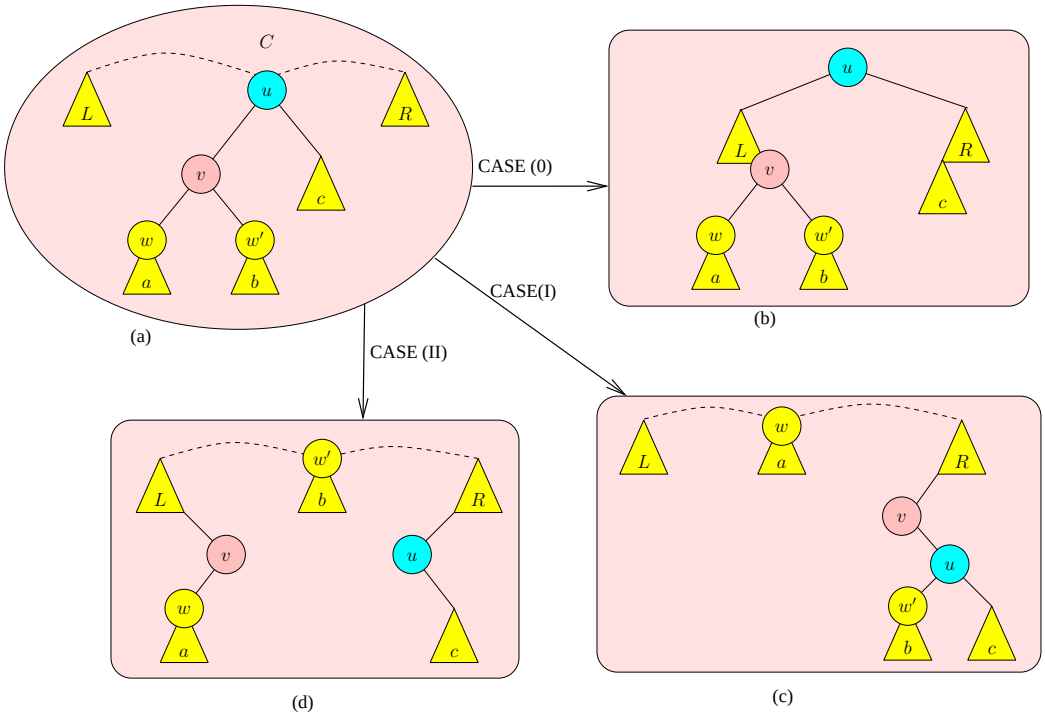


Figure 6: Top Splay: Cases 0, I and II

Here then, is the solution: we maintain 3 splay trees, L, C, R, corresponding to the Left-, Center- and Right-trees. Refer to Figure 6(a). Initially Left- and Right-trees are empty, and the Center-tree is the input tree. The keys in L are less than the keys in C, which are in turn less than the keys in R. Inductively, assume the key K that we are looking for is in C. There are three cases: suppose u is the root.

- CASE (0): This is the base case. The key K we are looking for is equal to the key at u. We attach the left and right children of u at the rightmost tip and leftmost tip of L and R, resp. We then make u the root of a tree with left and right children L and R. See Figure 6(b). In case K is equal to the key at a child of u, we just rotate that child before applying the preceding transformation.
- CASE (I): Suppose the key K is found in the subtree rooted at an outer grandchild w of u. By symmetry, assume that w is the left child of v, where v is the left child of u. In this case, we transfer the nodes u, v and the right subtrees of u and v to the left-tip of R, as shown in Figure 6(c).

- CASE (II): Suppose key K is found in the subtree rooted at an inner grandchild w' of u. By symmetry, assume w' is right child of v, where v is the left child of u. In this case, we transfer the node u and its right subtree to the left-tip of R, and the node v and its left-subtree to the right-tip of L, as shown in Figure 6(d).

The correctness of this procedure is left as an easy exercise.

EXERCISES

Exercise 2.1: Comment on the following assertions:
(a) Every splay tree is a binary search tree.
(b) Every binary search tree is a splay tree. ♢

Exercise 2.2: In this question, we ask which BSTs can arise from splay operations. Define an **IL splay tree** to be one that can be obtained by a sequence of insertions (I) and lookups (L). into an initially empty splay tree. Note that we omit deletes (D). If we include deletes, we have an **ILD splay tree**. Remember that it is the **shape** of a BST that matters, and there is no need to indicate explicit keys in your BSTs. Furthermore, *assume that insertion and deletion is implemented by first doing the standard BST insertion/deletion, followed by a splay.* This assumption will facilitate our enumeration of ILD shapes: For instance, given any shape of size n, it is easy to see all the shapes of size $n+1$ that can result from an insertion: just add a leaf, and splay from that leaf. There are $n+1$ ways to insert this leaf.
(a) Show that not every BST on 3 nodes is an IL splay tree.
(b) Show that every BST on 3 nodes is an ILD splay tree.
(c) Show that any BST on 4 nodes is an IL splay tree.
(d) Show that any BST on 5 nodes is an IL splay tree.
HINT: Furthermore, it is sufficient to enumerate shapes up to left-right symmetry (e.g., for BSTs with 3 nodes, there are only distinct 3 shapes, up to left-right symmetry). Moreover, when we insert into a given
(e) Prove or disprove: every BST on $n > 5$ nodes is an IL splay tree. ♢

Exercise 2.3: In the text, we splay the sequence of keys $n, n-1, n-2, \ldots, 1$ in the tree T_n to show that for naive splaying is not amortized $O(\log n)$. Please re-do this example using regular splaying. First illustrates what happens for $n = 5$ and $n = 6$. What is the end result after doing the sequence of lookups $n, n-1, \ldots, 2, 1$ on T_n. ♢

Exercise 2.4: Perform the following splay tree operations, starting from an initially empty tree.

$$\texttt{insert}(3,2,1;6,5,4;9,8,7), \texttt{lookUp}(3), \texttt{delete}(7), \texttt{insert}(12,15,14,13), \texttt{split}(8).$$

Show the splay tree after each step. ♢

Exercise 2.5: Show the result of $\texttt{merge}(T_1, T_2)$ where T_1, T_2 are the splay trees shown in Figure 7. ♢

Exercise 2.6: Consider the insertion of the following sequence of keys into an initially empty tree: $1, -1, 2, -2, 3, -3, \ldots, n, -n$. Let T_k be the splay tree after inserting k (for $k = 1, -1, 2, -2,$

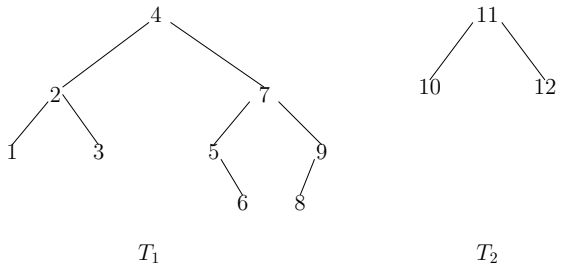


Figure 7: Splay trees T_1, T_2

etc).
(i) Show T_n and T_{-n} for $n = 1, 2, 3$.
(ii) State and prove a conjecture about the shape of T_n. ♢

Exercise 2.7: Consider the insertion of the following sequence of keys into an initially empty tree: $3, 2, 1; 6, 5, 4; 9, 8, 7; \ldots; 3n, 3n-1, 3n-2$, where we have written the keys in groups of three to indicate the pattern. Let T_n be the splay tree after the nth insertion (so T_0 is the empty tree).
(a) Show T_{3n} for $n = 1, 2, 3, 4$.
(b) State and prove a conjecture about the shape of T_n.
(a', b') Repeat (a) and (b), but use the following sequence of input keys

$$2, 3, 1; 5, 6, 4; 8, 9, 7; \ldots; 3n-1, 3n, 3n-2.$$

♢

Exercise 2.8: Consider the insertion of the following sequence of keys into an initially empty tree: $2, 3, 1; 5, 6, 4; 8, 9, 7; \ldots; 3n-1, 3n, 3n-2$, where we have written the keys in groups of three to indicate the pattern. Let T_n be the splay tree after the nth insertion (so T_0 is the empty tree).
(a) Show T_{3n} for $n = 1, 2, 3, 4$.
(b) State and prove a conjecture about the shape of T_n. ♢

Exercise 2.9: (Open ended) Prove that if we have any "regular" sequence of insertions, as in the previous exercises, the result is a "regular" splay tree. Part of this problem is to capture various notions of regularity. Let us capture the previous two exercises: let $G(n) = (g_1(n), g_2(n), \ldots, g_k(n))$ where k is a constant and $g_i(n)$ is a integer polynomial in n. E.g. $G(n) = (n, -n)$ and $G(n) = (3n, 3n-1, 3n-2)$ captures the regularity of the previous two exercises. Assume that the sequences $G(n)$ and $G(m)$ have different members for $m \neq n$. Then we want to show that the insertion of the sequence $G(1); G(2); G(3); \ldots; G(n)$ yields a splay tree T_n that is "regular", but in what sense? ♢

Exercise 2.10: Fix a key K and a splay tree T_0. Let $T_{i+1} \leftarrow \texttt{splay}(K, T_i)$ for $i = 0, 1, \ldots$.
(a) Under what conditions will the T_i's stabilize (become a constant)? How many splays will bring on the stable state?
(b) Under what conditions will the T_i's not stabilize? What is the ultimate condition? (Describe as thoroughly as you can) How many splays will bring on this condition? ♢

Exercise 2.11: Let T be a binary search tree in which every non-leaf has one child. Thus T has a linear structure with a unique leaf.

(a) What is the effect of `lookUp` on the key at the leaf of T?
(b) What is the minimum number of `lookUp`'s to make T balanced? ♢

Exercise 2.12: In our operations on splay trees, we usually begin by performing a splay. This was not the case with our insertion algorithm in the text. But consider the following variant insertion algorithm. To insert an item X into T:
1. Perform `splay`$(X.\texttt{key}, T)$ to give us an equivalent tree T'.
2. Now examine the key K' at root of T': if $K' = X.\texttt{key}$, we declare an error (recall that keys must be distinct).
3. If $K' > X.\texttt{key}$, we install a new root containing X, and K' becomes the right child of X; the case $K' < X.\texttt{key}$ is symmetrical. In either case, the new root has key equal to $X.\texttt{key}$. See Figure 8.
(a) Prove that the amortize complexity this insertion algorithm remains $O(\log n)$.
(b) Compare the amortized complexity of this method with the one in the text. Up to constant factors, there is no difference, of course. But which has a better constant factor? ♢

Exercise 2.13: As in the previous Exercise, carry out the details of the variant deletion algorithm noted in the text. ♢

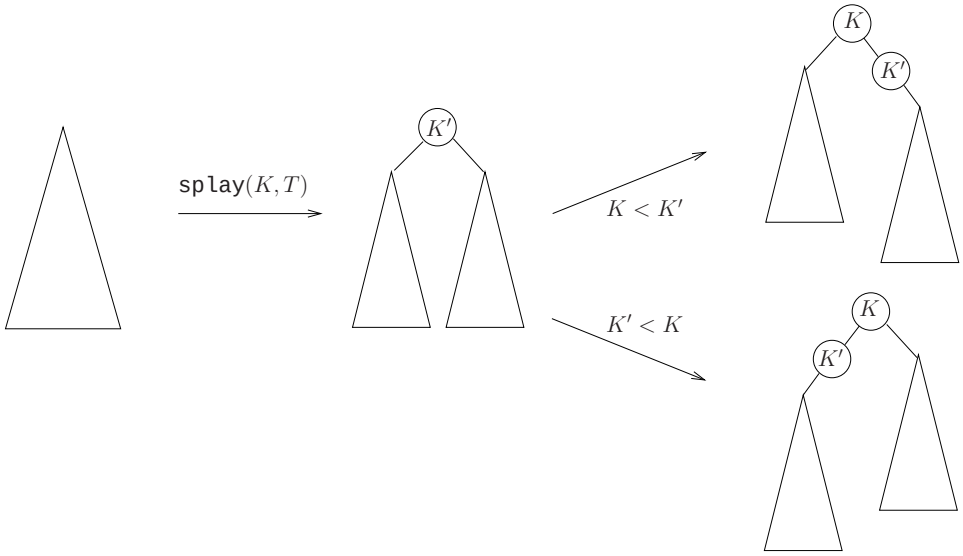


Figure 8: Alternative method to insert a key K.

Exercise 2.14: (Top Down Splaying)
(a) What is the "pitfall" mentioned for the initial implementation of the top-down splaying algorithm.
(b) Show that the amortized complexity of the second attempt cannot be $O(\log n)$. To be specific, here is the code:

```
topSplay(K, u)
    Input: K is a key and u a node; we are looking for K in the subtree T_u rooted at u
    Output: We return "found" or "not found". In any case, as side effect, in place of u
            will be a node containing the predecessor or successor of K in the tree T_u.
1.  If (u.key = K) Return("found").
2.  case(State)
        State=0:
            If (u.key > K)
                rotate(u.left);
                State ← 2.
            else
                rotate(u.right);
                State ← 1.
        State=1:
            If (u.key > K)
                v ← u.left.right;
                If (v = nil) Return("not found").
                rotate²(v);
                State ← 3;
            else
                v ← u.right;
                If (v = nil) Return("not found").
                rotate(v);    ◁ Remain in State 1.
        State=2:
            ···    ◁ Omitted: symmetrical to State 1.
        State=3:
            If (u.key > K)
                v ← u.left.right
            else
                v ← u.right.left.
            If (v = nil) Return("not found").
            rotate²(v)
      ◁ End Case Statement
3.  topSplay(K, v).    ◁ Tail recursion
```

♢

Exercise 2.15: A **splay tree** is a binary search tree that arises from a sequence of insertions, lookups and deletes, starting from empty trees.

(a) Is every binary search tree a splay tree?

(b) Let T and T' be equivalent binary search trees (i.e., they store the same set of keys). Can we transform T to T' by repeated splays? ♢

END EXERCISES

§3. Splay Analysis

Our main goal is to prove:

THEOREM 1 (Splay Theorem). *The amortized cost of each splay operation is $O(\log n)$ assuming at most n items in a tree.*

Before proving this result, let us show that it is a "true" amortization bound. More precisely, we show that the worst case bound is linear, is not logarithmic. To see this, consider the repeated insertion of the keys $1, 2, 3, \ldots, n$ into an initially empty tree. In the ith insertion, we insert key i. We claim that this results in a tree L_n that is just a linear list: in general, the root of L_n contains the key n, with left subtree L_{n-1} and empty right subtree. See Figure 3(e) for L_5. If we next insert key $n+1$ into L_n, it will be the right child of the root, but upon splaying, the newly inserted key $n+1$ becomes the root. Thus our induction is preserved. Finally, if we perform a lookup on key 1 in this tree, we must expend $\Theta(n)$ units of work.

¶8. Proof of Splay Theorem. To start the amortized analysis, we must devise a potential function: let $\text{SIZE}(u)$ denote, as usual, the number of nodes in the subtree rooted at u. Define its **potential** to be

$$\Phi(u) = \lfloor \lg \text{SIZE}(u) \rfloor .$$

Note that $\text{SIZE}(u) = 1$ iff u is a leaf. Thus $\Phi(u) = 0$ iff u is a leaf. If $S = \{u_1, u_2, \ldots u_k\}$ is a set of nodes, we may write $\Phi(S)$ or $\Phi(u_1, u_2, \ldots u_k)$ for the sum $\sum_{u \in S} \Phi(u)$. If S is the set of nodes in a splay tree T or in the entire data structure then $\Phi(S)$ is called the potential of (respectively) T or the entire data structure. By definition, a tree with 0 or 1 node has no potential, $\Phi(T) = 0$.

LEMMA 2 (Key Lemma). *Let Φ be the potential function before we apply* `splayStep`*(u), and let Φ' be the potential after. The credit-potential invariant is preserved if we charge the SplayStep*

$$3(\Phi'(u) - \Phi(u)) \tag{4}$$

units of work in cases I and II. In the base case, we charge one extra unit, in addition to the charge (4).

Theorem 1 follows easily from the Key Lemma. To see this, suppose that splaying at u reduces to a sequence of k SplaySteps at u and let $\Phi_i(u)$ be the potential of u after the ith SplayStep. The total charges to this sequence of SplaySteps is

$$1 + \sum_{i=1}^{k} 3[\Phi_i(u) - \Phi_{i-1}(u)] = 1 + 3[\Phi_k(u) - \Phi_0(u)]$$

by telescopy. Note that the "1" comes from the fact that the last SplayStep may belong to the base case. Clearly this total charge is at most $1 + 3 \lg n$. To finish off the argument, we must account for the cost of looking up u. But it easy to see that this cost is proportional to k and so it is proportional to the overall cost of splaying. This only increases the constant factor in our charging scheme. This concludes the proof of the main goal.

¶9. Proof of Key Lemma. The following is a useful remark about rotations:

LEMMA 3. *Let Φ be the potential function before a rotation at u and Φ' the potential function after. Then the increase in potential of the overall data structure is at most*

$$\Phi'(u) - \Phi(u).$$

The expression $\Phi'(u) - \Phi(u)$ is always non-negative.

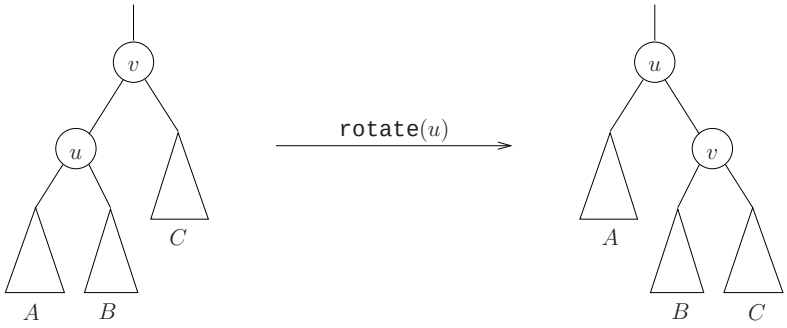


Figure 9: Rotation at u.

Proof. We refer to Figure 9. The increase in potential is

$$\begin{array}{rcll} \Delta\Phi & = & \Phi'(u,v) - \Phi(u,v) & \\ & = & \Phi'(v) - \Phi(u) & (\text{as } \Phi'(u) = \Phi(v)) \\ & \leq & \Phi'(u) - \Phi(u) & (\text{as } \Phi'(u) \geq \Phi'(v)). \end{array}$$

It is obvious that $\Phi'(u) \geq \Phi(u)$. **Q.E.D.**

Proof of Key Lemma. The Base Case is almost immediate from lemma 3: the increase in potential is at most $\Phi'(u) - \Phi(u)$. This is at most $3(\Phi'(u) - \Phi(u))$ since $\Phi'(u) - \Phi(u)$ is non-negative. The charge of $1 + 3(\Phi'(u) - \Phi(u))$ can therefore pay for the cost of this rotation and any increase in potential.

Refer to Figure 4 for the remaining two cases. Let the sizes of the subtrees A, B, C, D be a, b, c, d, respectively.

Consider Case I. The increase in potential is

$$\begin{array}{rcll} \Delta\Phi & = & \Phi'(u,v,w) - \Phi(u,v,w) & \\ & = & \Phi'(v,w) - \Phi(u,v) & (\text{as } \Phi'(u) = \Phi(w)) \\ & \leq & 2(\Phi'(u) - \Phi(u)) & (\text{as } 2\Phi'(u) \geq \Phi'(v,w), \quad 2\Phi(u) \leq \Phi(u,v)). \end{array}$$

Since $\Phi'(u) \geq \Phi(u)$, we have two possibilities: (a) If $\Phi'(u) > \Phi(u)$, then the charge of $3(\Phi'(u) - \Phi(u))$ can pay for the increased potential *and* the cost of this splay step. (b) Next suppose $\Phi'(u) = \Phi(u)$. By assumption, $\Phi'(u) = \lfloor \lg(3 + a + b + c + d) \rfloor$ and $\Phi(u) = \lfloor \lg(1 + a + b) \rfloor$ are equal. Thus $1 + a + b > 2 + c + d$, and so $3 + a + b + c + d > 2(2 + c + d)$ and

$$\Phi'(w) = \lfloor \lg(1 + c + d) \rfloor < \lfloor \lg(3 + a + b + c + d) \rfloor = \Phi(u).$$

Also,

$$\Phi'(v) \leq \Phi'(u) = \Phi(u) \leq \Phi(v).$$

Combining these two inequalities, we conclude that

$$\Phi'(w,v) < \Phi(u,v).$$

Hence $\Delta\Phi = \Phi'(w,v) - \Phi(u,v) < 0$. Since potentials are integer-valued, this means that $\Delta\Phi \leq -1$. Thus the change in potential releases at least one unit of work to pay for the cost of the splay step. Note that in this case, we charge nothing since $3(\Phi'(u) - \Phi(u)) = 0$. Thus the credit-potential invariant holds.

Consider Case II. The increase in potential is again $\Delta\Phi = \Phi'(v,w) - \Phi(u,v)$. Since $\Phi'(v) \leq \Phi(v)$ and $\Phi'(w) \leq \Phi'(u)$, we get

$$\Delta\Phi \leq \Phi'(u) - \Phi(u).$$

If $\Phi'(u) - \Phi(u) > 0$, then our charge of $3(\Phi'(u) - \Phi(u))$ can pay for the increase in potential and the cost of this splay step. Hence we may assume otherwise and let $t = \Phi'(u) = \Phi(u)$. In this case, our charge is $3(\Phi'(u) - \Phi(u)) = 0$, and for the credit potential invariant to hold, it suffices to show

$$\Delta\Phi < 0.$$

It is easy to see that $\Phi(v) = t$, and so $\Phi(u, v) = 2t$. Clearly, $\Phi'(v, w) \leq 2\Phi'(u) = 2t$. If $\Phi'(v, w) < 2t$, then $\Delta\Phi = \Phi'(v, w) - \Phi(u, v) < 0$ as desired. So it remains to show that $\Phi'(v, w) = 2t$ is impossible. For, if $\Phi'(v, w) = 2t$ then $\Phi'(v) = \Phi'(w) = t$ (since $\Phi'(v), \Phi'(w)$ are both no larger than t). But then

$$\Phi'(u) = \left\lfloor \lg(\text{SIZE}'(v) + \text{SIZE}'(w) + 1) \right\rfloor \geq \left\lfloor \lg(2^t + 2^t + 1) \right\rfloor \geq t + 1,$$

a contradiction. Here, SIZE$'$ denotes the size after the splay step operation. This proves the Key Lemma.

LEMMA 4. *Let T' be a binary tree with $n + 1$ items, and T is obtained from T' by deleting some leaf x. Then $\Phi(T') - \Phi(T) \leq \lg n$.*

Proof. Let $(u_0, u_1, \ldots, u_m)$ denote the path from the root of T' to $x = u_m$. The potential of each u_i is $\lfloor \lg(n_i + 1) \rfloor$ where $n > n_0 > n_1 > \cdots > n_m = 0$. Furthermore,

$$\Phi(T') - \Phi(T) = \sum_{i=0}^{m-1} \lfloor \lg(n_i + 1) \rfloor - \lfloor \lg(n_i) \rfloor .$$

But we observe that $\lfloor \lg(n_i + 1) \rfloor - \lfloor \lg(n_i) \rfloor = 1$ iff $n_i + 1$ is a power of 2. There are at most $\lg n$ values of i for which this happens. Hence

$$\Phi(T') - \Phi(T) \leq \lg n.$$

Q.E.D.

¶10. Amortized Cost of Splay Implementation of ADT Operations. We conclude with an amortized cost statement for splay trees.

THEOREM 5. *Starting from an initially empty splay tree, any sequence of m requests of the types*

`lookUp`, `insert`, `merge`, `delete`, `deleteMin`, `split`,

and involving a total of n items, has total time complexity of $O(m \log n)$. Thus, the amortized cost is $\log n$ per request.

Proof. This follows almost immediately from Theorem 1 since each request can be reduced to a constant number of splay operations plus $O(1)$ extra work. The splay operations are charged $O(\lg n)$ units, and the extra work is charged $O(1)$ units. But two important details must not be overlooked: sometimes, the extra $O(1)$ work increases the potential by a non-constant amount, and this increase must be properly charged. This situation happens in two situations.

(A) When inserting a new key: the key will become a leaf of the tree, followed by splaying at this leaf. While the splaying cost is accounted for, the act of creating this leaf may also increase the potential of every node along the path to the leaf. By the previous lemma, this increase is at most $\lg n$.

(B) When merging two trees T_1, T_2. In this case, we first perform `splay`$(+\infty, T_1)$. If u is the root of the resulting tree, then u has no right child and we simply attach T_2 as the right subtree of u. This "attachment" will increase the potential of u by less than $1 + \lg(1 + |T_2|/|T_1|) < 1 + \lg n$. Thus we just have to charge this operation an extra $1 + \lg n$ units. Note that deletion is also reduced to merging, and

so its charge must be appropriately increased. In any case, all charges remain $O(\lg n)$, as we claimed. **Q.E.D.**

Note that the above argument does not immediately apply to `topSplay`: this is treated in the Exercises. Sherk [9] has generalized splaying to k-ary search trees. In such trees, each node stores an ordered sequence of $t-1$ keys and t pointers to children where $2 \leq t \leq k$. This is similar to B-trees.

¶11. Application: Splaysort Clearly we can obtain a sorting algorithm by repeated insertions into a splay tree, followed by repeated deleteMins — this is analogous to the heapsort algorithm (Lect.III,¶7). This algorithm is known as **splaysort**. It is not only theoretically optimal with $O(n \log n)$ complexity, but has been shown to be quite practical [6]. Splaysort has the ability to take advantage of "presortedness" in the input sequence. For instance, running splaysort on the input sequence $x_1 > x_2 > \cdots > x_n$ will take only $O(n)$ time in the worst case. One way to quantify presortedness is to count the number of pairwise inversions in the input sequence. E.g., if the input is already in sorted order, we would like to (automatically) achieve an $O(n)$ running time. However, if $x_1 < x_2 < \cdots < x_n$ is our input, the advantage of presortedness is lost for our version of splaysort. The Exercises discuss some ways to overcome this.

Quicksort (Lectures II and VIII) is regarded as one of the fastest sorting algorithms in practice. But algorithms like splaysort may run faster than Quicksort for "well presorted" inputs. Quicksort, by its very nature, deliberately destroy any property such as presortedness in its input.

EXERCISES

Exercise 3.1: Where in the proof is the constant "3" actually needed in our charge of $3(\Phi'(u) - \Phi(u))$? ♢

Exercise 3.2: Adapt the proof of the Key Lemma to justify the following variation of SplayStep:

> VARSPLAYSTEP(u):
> (Base Case) If u is a child or grandchild of the root,
> then rotate once or twice at u until it becomes the root.
> (General Case) else rotate at u.`parent`, followed by two rotations at u.

♢

Exercise 3.3: Let us define the potential of node u to be $\Phi(u) = \lg(\text{SIZE}(u))$, instead of $\Phi(u) = \lfloor \lg(\text{SIZE}(u)) \rfloor$.

(a) How does this modification affect the validity of our Key Lemma about how to charge `splayStep`? In our original proof, we had 2 cases: either $\Phi'(u) - \Phi(u)$ is 0 or positive. But now, $\Phi'(u) - \Phi(u)$ is always positive. Thus it appears that we have eliminated one case in the original proof. What is wrong with this suggestion?

(b) Consider Case I in the proof of the Key Lemma. Show that if $\Phi'(u) - \Phi(u) \leq \lg(6/5)$ then $\Delta\Phi = \Phi'(w,v) - \Phi(u,v) \leq -\lg(6/5)$. HINT: the hypothesis implies $a + b \geq 9 + 5c + 5d$.

(c) Do the same for Case II. ♢

Exercise 3.4: Develop amortized versions of the rotation-based deletion and successor algorithms from Lecture III. ♢

Exercise 3.5: We continue with the development of the previous question; in particular, we still define $\Phi(u)$ to be $\lg \text{SIZE}(u)$.
(i) Let T be a splay tree on n nodes, and let T' be the result of inserting a new key into T using the standard insertion algorithm. So, the new key appears as a leaf u in T'. Prove that $\Phi(T') - \Phi(T) = O(\lg n)$.
(ii) Prove the Key Lemma under this new definition of potential: *i.e.*, it suffices to charge $3(\Phi'(u) - \Phi(u))$ units for Case I and II of `splayStep`. HINT: Although $\Phi'(u) - \Phi(u)$ is always positive, we need to ensure that $\Phi'(u) - \Phi(u) \geq \alpha$ for some fixed positive constant α. So you must prove that in case $\Phi'(u) - \Phi(u) < \alpha$, the increase in potential, $\Delta\Phi$, is actually a negative value less than $-\alpha$. You may choose $\alpha = \lg(5/4)$.
(iii) Conclude with the alternative proof of the main theorem on splay trees (Theorem 1). ♢

Exercise 3.6:
(i) Is it true that splays always decrease the height of a tree? The average height of a tree? (Define the average height to be the average depth of the leaves.)
(ii) What is the effect of splay on the last node of a binary tree that has a linear structure, *i.e.*, in which every internal node has only one child? HINT: First consider two simple cases, where all non-roots is a left child and where each non-root is alternately a left child and a right child. ♢

Exercise 3.7: Assume that node u has a great-grandparent. Give a simple description of the effect of the following sequence of three rotations: `rotate`(u.`parent.parent`); `rotate`(u.`parent`); `rotate`(u). ♢

Exercise 3.8: Does our Key Lemma hold if we define $\Phi(u) = \lceil \lg \text{SIZE}(u) \rceil$? ♢

Exercise 3.9: For any node u,

$$\Phi(u_L) = \Phi(u_R) \Rightarrow \Phi(u) = \Phi(u_L) + 1$$

where u_L, u_R are the left and right child of u. ♢

Exercise 3.10: Modify our splay trees to maintain (in addition to the usual children and parent pointers) pointers to the successor and predecessor of each node. Show that this can be done without affecting the asymptotic complexity of all the operations (`lookUp`, `insert`, `delete`, `merge`, `split`) of splay trees. ♢

Exercise 3.11: We consider some possible simplifications of the `splayStep`.
(A) One-rotation version: Let `splayStep`(u) simply amount to `rotate`(u).
(B) Two-rotation version:

> SPLAYSTEP(u):
> (Base Case) If u.`parent` is the root, `rotate`(u).
> (General Case) else do `rotate`(u.`parent`), followed by `rotate`(u).

For both (A) and (B):
(i) Indicate how the proposed SplayStep algorithm differs from the original.
(ii) Give a general counter example showing that this variation does not permit a result similar to the Key Lemma. ♢

Exercise 3.12: Modify the above algorithms so that we allow the search trees to have identical keys. Make reasonable conventions about semantics, such as what it means to lookup a key. ♢

Exercise 3.13: Consider the `topSplay` algorithm:
(i) Characterize the situations where it gives the same result as the 2-pass `splay` algorithm.
(ii) OPEN: Does it have amortized cost of $O(\log n)$? ♢

Exercise 3.14: Consider the following variant for a potential function. For any node u in a splay tree, let $\Phi(u)$ be defined to be H_n where n is the size of the subtree rooted at u. Prove the analogue of our main theorem. ♢

Exercise 3.15: Repeat the previous question, but define $\Phi(u)$ to be $\lg n$ (as opposed to the normal definition, $\Phi(u) = \lfloor \lg n \rfloor$). ♢

END EXERCISES

§4. Application to Dynamic String Encoding

We present a remarkable application of splay trees to the problem of dynamic Huffman code of Chapter V. Recall that in this problem we want a protocol to transmit any string $X \in \Sigma^*$ over an alphabet Σ. The transmission protocol is restricted to make only one pass over the string X and as each letter in X is encountered, the transmission protocol emits a binary sequence. The receiver protocol, on receiving this binary sequence can uniquely reconstruct the corresponding letter. It is assumed that $\Sigma \subseteq \{0,1\}^N$ where the set Σ is known to the receiver. The usual example is where Σ is the extended ASCII set and $N = 8$. Of course, a naive protocol is to just emit N bits for each $x \in \Sigma$; so the goal is to design a protocol that transmits as few bits as possible, presumably taking advantage of the nature of X. For instance, transmitting $X =$ `aaaaaa` should be cheaper that transmitting $X =$ `abcdef`.

We now need to use our splay tree T as an external binary search tree: recall from Chapter III that this means only the leaves of T store items. The internal nodes do not store items, but store keys which are used to guide the lookup or search process. Note that keys of items must be unique, but two or more internal nodes could hold the same key, and these keys generally come from keys of some inserted item. The keys in such a splay tree has the usual BST property: for any internal node u, we have

$$u_L.\mathsf{key} < u.\mathsf{key} \leq u_R.\mathsf{key}$$

where u_L (u_R) is any node in the left (right) subtree of u. This is similar to the idea of (a,b)-search trees. our splaytree algorithms for lookup, insertion and deletion can be modified appropriately. For the current purposes, we only need to consider the problem of insertion.

To insert an item with key k, we do the usual lookup on k. This process will end up at some leaf node x. There are now two possibilities:

- (Insertion failure) If $x.\mathsf{key} = k$ then we cannot insert. But we still need to splay in order to amortize the cost. If x has a parent p, we will splay p (not x).

- (Insertion success) If $x.\texttt{key} \neq k$, we can create a new leaf node y in order to hold the inserted item. We also create a new internal node u which will become the parent of x and y. The BST property requires that the left (right) child of u will store the smaller (larger) of $x.\texttt{key}$ and $y.\texttt{key}$. We also need to store a key in the node u: this key can be chosen to be the key in the right child of u. If x did not have a parent, then u is the new root and we are done. Otherwise, let p be the (former) parent x. We make u the child of p in the place of x. Finally, we must splay u.

So, this is the source of duplicate keys in internal nodes!

Observe that we always splay internal nodes: we are not allowed to splay leaf nodes.

For our application of compressing strings over the alphabet Σ, we will use Σ as our universe of keys, assumed to have some sorting order. Our splay tree T will represent store a subset $S \subseteq \Sigma$ in its leaves. We can view T as a prefix-free code for S: the path from the root of T to a letter $x \in S$ can be read as a binary string which is the prefix-free code for x. As in dynamic Huffman codes, we require every splay tree T to have a leaf representing a special key $*$, where $*$ is not a letter in Σ. In our sorting order, we assume $*$ is less than every letter in Σ. So the smallest possible splay tree has one node (which is both root and leaf) storing $*$. Call this the $*$-tree.

Let us show the sequence of trees that results from insertion of consecutive letters from our famous 12-character string: `hello␣world!` Here is the sorting order we use in this example:

$$\texttt{*} < \texttt{a} < \texttt{b} < \texttt{c} < \cdots < \texttt{x} < \texttt{y} < \texttt{z} < \sqcup < \texttt{!}$$

Initially, we have the $*$-tree.

FIGURE/EXAMPLE...

The dynamic Huffman tree solution amounts to maintaining a Huffman tree that is constantly adjusting its shape as the frequency function changes. We will imitate this, except that we now use a splay tree T instead of a Huffman tree. Assume $X = x_1 x_2 \ldots x_m$, and we have scanned the first i letters of X. Then the splay tree T will have $1 + |\{x_1, x_2, \ldots, x_i\}|$ external nodes, one for each letter in the set $\{x_1, \ldots, x_i\} \subseteq \Sigma$, including a special symbol $*$ that is distinct from Σ. We assume a total ordering on $\Sigma \cup \{*\}$ where $*$ is less than any $a \subseteq \Sigma$. For $a \in \Sigma \cup \{*\}$, let the a**-path** refers to the bit string encoding the path from root to the external node associated with a. If a is not in any external node, then the a**-path** refers to the $*$-path.

We now address the transmission protocol. Upon reading the next letter x_{i+1} in X, we we perform a standard lookup on x_{i+1}. The lookup path will end in some external node b. If $b.\texttt{key} = x_{i+1}$, then we simply emit the bits that encode the path from the root to b. If $b.\texttt{key} \neq x_{i+1}$, then we emit the $*$-path, followed by the N bits in the standard encoding of x_{i+1}. Recall that we have to create a new internal node u and also a new external node c. We store x_{i+1} in c, and make b and c the children of u, such that the left child of u has key less than the key at right child. Moreover, we can duplicate the key of the right child and put this key in u. Thus T is now a BST. We splay u.

Now it is quite clear how the receiving protocol maintains its own copy of the splay tree T, and mirrors the action of the transmitter exactly. In particular, the receiving protocol can reconstruct the standard letter x_{i+1} from what is emitted at each step: if the emitted It is clear that what is emitted by the transmission protocol is self-limiting in the sense that the receiving protocol knows when the emission for a letter has ended.

Let us illustrate this for the famous 12-letter string, `hello world!`.

I should introduce the external BST in Chap 3 and do the hello world example there...

§5. Application to Convex Hulls

The following application is interesting because it illustrates the idea of an **implicit binary search tree**. The usual notion of keys is inapplicable. But by using information distributed at a node u and its children u_L and u_R, we are able to perform tests to simulate searching in a binary search tree.

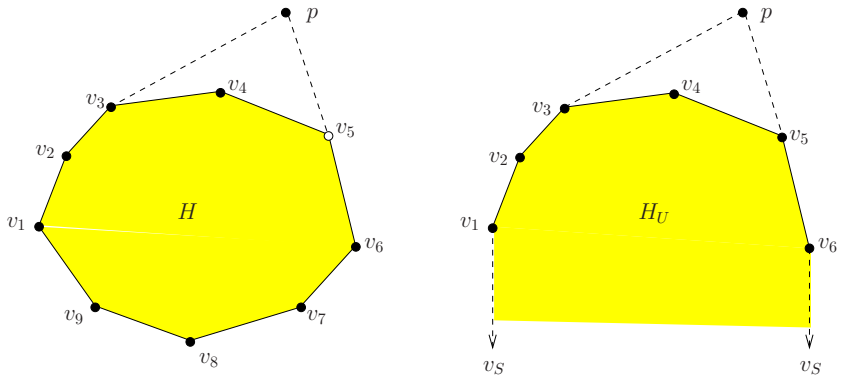
Given a set X of $n \geq 1$ points in the plane, its **convex hull** $CH(X)$ is the smallest convex subset of the plane that contains X. As the boundary of $CH(X)$ is a convex polygon, we may represent it as a sequence

$$H = (v_1, v_2, \ldots, v_m), \quad 1 \leq m \leq n$$

where $v_i \in X$ and these v_i's appear as consecutive vertices in a clockwise traversal of the polygon $CH(X)$. We assume that H is a strictly convex sequence, i.e., no 3 consecutive vertices v_{i-1}, v_i, v_{i+1} are collinear. We shall use H and $CH(X)$ interchangeably. We want to dynamically maintain H subject to two types of requests:

$$\texttt{tangent}(p, H) \quad \text{and} \quad \texttt{insert}(p, H)$$

where p is an arbitrary point. If p is outside H, $\texttt{tangent}(p, H)$ will return a pair (q, r) of distinct points on H such that the lines $\overline{pq}$ and $\overline{pr}$ are both tangential to H. Call q and r the **tangent points** of H from p. We will specify an ordering on q, r later so that q (resp., r) is the left (resp., right) tangent points. E.g., in Figure 10(a), $\texttt{tangent}(p, H)$ returns (v_3, v_5).



(a) (b)

Figure 10: (a) $H = (v_1, \ldots, v_9)$, (b) $H_U = (v_1, \ldots, v_6)$.

If p is inside the current hull, H has no tangent points from p, and we return "↑" The request $\texttt{insert}(p, H)$ means we want to update H to represent $CH(X \cup \{p\})$. Note that if p is inside the current hull, H is unchanged.

¶12. Reduction to Half-Hulls. We may assume that v_1 and v_ℓ ($1 \leq \ell \leq m$) have (resp.) the smallest and largest x-coordinates among the v_i's. For simplicity, assume that any two consecutive vertices, v_i and v_{i+1}, have distinct x-coordinates so that v_1 and v_ℓ are unique. Then we can break H into two convex chains,

$$H_U = (v_1, v_2, \ldots, v_\ell), \qquad H_L = (v_1, v_m, v_{m-1}, \ldots, v_{\ell+1}, v_\ell).$$

So H_U and H_L share precisely their common endpoints. Assuming that H_U lies above the segment $v_1 v_\ell$, we call H_U the **upper hull** and H_L the **lower hull** of H. Let $v_S = (0, -\infty)$ and $v_N = (0, +\infty)$

be points[4] at infinity (the South and North Poles, respectively). In a certain sense, H_U is the convex hull of $X \cup \{v_S\}$:

$$CH(X \cup \{v_S\}) = (v_1, v_2, \ldots, v_\ell, v_S).$$

This can be made precise by a limiting argument. Similarly, H_L is the convex hull of $X \cup \{v_N\}$. Collectively, H_U and H_L are the two **half-hulls** of X. We implement $\texttt{insert}(p, H)$ by reducing it to insertion into the upper and lower hulls:

$$\texttt{insert}(p, H_U); \texttt{insert}(p, H_L).$$

By symmetry, we may focus on upper hulls. Clearly p is inside H iff p is inside both H_U and H_L. Now suppose p is not inside $H_U = (v_1, \ldots, v_\ell)$. Then $\texttt{tangent}(p, H_U)$ returns the pair (q, r) of tangent points of H_U from p where q lies to the left of r. For instance, $\texttt{tangent}(p, H_U)$ returns v_3, v_5 in Figure 10. There are two special cases. If p is left of v_1, then $q = v_S$; if p is right of v_ℓ then $r = v_S$. The details of how to reduce $\texttt{tangent}(p, H)$ to half-hulls is left to an exercise.

¶13. Reduction to Fully Mergeable Dictionary Operations. We are now going to store the upper hull H_U in a binary search tree T using the x-coordinates of vertices as keys. Suppose the sequence of points in H_U is $(v_1, \ldots, v_\ell)$ sorted so that

$$v_1 <_x v_2 <_x \cdots <_x v_\ell, \tag{5}$$

where, in general, we write

$$p <_x q \tag{6}$$

for points p and q such that $p.x < q.y$. Similarly, we may write $p \leq_x q$, $p =_x q$, $p <_y q$, etc.

To facilitate the insertion of new points, we must be able to split and merge our binary search trees. To see this, suppose we insert a new point p into H_U. Let $\texttt{tangent}(p, H_U) = (q, r)$ where $q <_x r$; we call q the **left tangent point** and r the **right tangent point**. Wlog, let $q = v_{i-1}$ and $r = v_{j+1}$); note that $i \leq j + 1$. So we need to replace a subsequence $(v_i, v_{i+1}, \ldots, v_j)$ $(1 \leq i \leq j \leq \ell)$ by the point p. Of course, if $i = j + 1$, the subsequence is empty. This replacement can be done efficiently if T is a splay tree that implements the operations of the fully mergeable dictionary ADT (¶5). By calling the split operation twice, we can split the upper hull into three upper hulls, $T_1 : (v_1, \ldots, v_{i-1})$, $T_2 : (v_i, \ldots, v_j)$ and $T_3 : (v_{j+1}, \ldots, v_\ell)$. Finally, we obtain the new upper hull by[5] forming the tree rooted at p with T_1 and T_3 as left and right subtrees.

We may assume that nodes in T have successor and predecessor pointers. We next show how to implement the requests

$$\texttt{insert}(p, T) \quad \text{and} \quad \texttt{tangent}(p, T)$$

where the binary tree T represents an upper hull. Note that both $\texttt{insert}(p, T)$ and $\texttt{tangent}(p, T)$ must take a common first step to decide if p in outside the upper hull of T or not. If not, then both algorithms returns $\uparrow$. So our first task is:

¶14. Deciding if a point p in outside a upper hull. This calls for an explicit binary search. Our first task is to determine the two consecutive hull vertices v_k, v_{k+1} such that

$$v_k <_x p \leq_x v_{k+1} \qquad (0 \leq k \leq \ell). \tag{7}$$

[4] We could define v_S and v_N to be $(r, -\infty)$ and $(r, +\infty)$, respectively, for any finite value r.

[5] Observe that this operation resembles a merge but is not the same as a standard merge. Nevertheless, an amortized cost of $O(\log n)$ can be achieved using our usual arguments.

To do this, we do a lookup on the key $p.x$. Recall that lookup will return u that is either the successor or predecessor of $p.x$ in T. From u, and using the successor and predecessor pointers in T, we can easily determine the k that satisfies (7).

By convention, if $k = 0$ then v_k is undefined, and if $k = \ell$ then v_{k+1} is undefined. Moreover either $k = 0$ or $k = \ell$ implies that p is outside the upper hull. Hence assume that $0 < k < \ell$. Now we conclude that p is outside the upper hull iff p lies above the line $\overline{v_k, v_{k+1}}$.

We now address the problem of computing $\texttt{tangent}(p, T)$, given that p lies outside the upper hull of T. We next give two methods for doing this.

¶15. Method One for Tangent Point: Walking Method. The first method is more intuitive. To implement $\texttt{tangent}(p, T)$, we have to find the left and right tangent points separately. By symmetry, let us only discuss the left tangent point.

Using the explicit binary search in ¶14, we can decide if p is outside the upper hull. If not, we can return $\uparrow$ since $\texttt{tangent}(p, T) = \uparrow$. Otherwise, this procedure gives us the k such that (7) holds. If $k = 0$, we can return $\uparrow$ since the left-tangent is undefined. Otherwise, we continue to search for the left tangent point q. We know that $q = v_{i_0}$ for some $i_0 \leq k$. To find i_0, we use the predecessor pointers to "walk" along the upper hull, starting from v_k to v_{k-1}, v_{k-2}, etc. In general, for any index i with the property that $v_i <_x p$, we can decide whether $i_0 = i$, $i_0 < i$ or $i_0 > i$ according to the following cases:

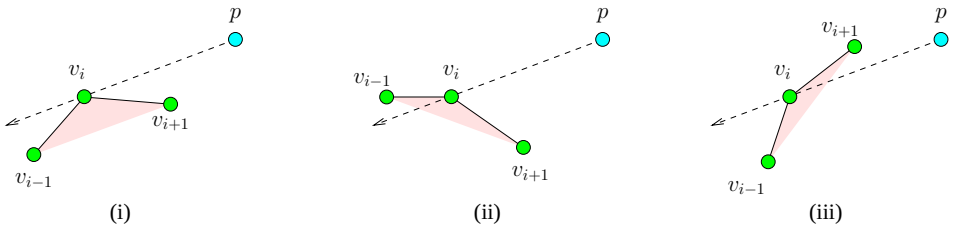


Figure 11: LeftTangent Search from p: (i) $i_0 = i$, (ii) $i_0 < i$, (iii) $i_0 > i$.

CASE (i) v_{i-1} and v_{i+1} lie below $\overline{v_i p}$.
Then $i_0 = i$. Here, if $i = 1$ then v_0 is the south pole (which lies below any line).

CASE (ii) v_{i+1}, but not v_{i-1}, lies below $\overline{v_i p}$.
Then $i_0 < i$.

CASE (iii) v_{i-1}, but not v_{i+1}, lies below $\overline{v_i p}$.
Then $i_0 > i$.

These three cases are illustrated in Figure 11. Of course, in our "walk" from v_k to v_{k-1}, v_{k-2}, etc, we will never encounter case (iii). We have ignored degeneracies in these three cases. This will be treated in the Exercises.

¶16. Method Two: Implicit Binary Search. The previous method for finding tangent points (q, r) takes linear time in the worst case. But if we were inserting the point p into the upper hull, then the elements lying strictly between q and r would be deleted. It is then easy to see that, in an amortized sense, the this linear search has amortized cost of $O(1)$. But if we view the convex hull as a generic

structure that supports arbitrary tangent queries, then this linear cost cannot be amortized to a sublinear cost. Our second method is able to overcome this problem and achieve an amortized cost of $O(\lg n)$.

Again, assume we want to find the left-tangent q (if it exists) of a query point p. As before, we call the explicit binary search in ¶14 to determine the k satisfying (7); we may assume that $k > 0$ and p is outside the upper hull (otherwise q does not exist). We now search for q *using another binary search*, not by walking along the upper hull. This search begins at the root of the splay tree T that stores the upper hull, and uses the 3-way decision (Cases (i)-(iii) above) to search for the left tangent point $q = v_{i_0}$. This is an "implicit" binary search because the 3-way decisions are not based on any explicitly stored keys. Indeed, the decisions depend not just on the stored data v_{i-1}, v_i, v_{i+1} but *also* the query point p. We are searching for the index i_0, but we use the data v_{i-1}, v_i, v_{i+1}, p to decide if $i = i_0$, or $i < i_0$ or $i > i_0$.

FINDLEFTTANGENTPOINT(p, T):
▷ *Explicit Binary Search*
1. Perform an explicit binary search using the key $p.x$.
   This locates the k such that (7) holds.
   We can then determine if p lies inside the upper hull.
   If so, Return($\uparrow$)
▷ *Implicit Binary Search*
   Initialize u to the root of T; let v_i be the vertex at u.
2. Repeat:
   If $v_i \geq_x p$, set $u \leftarrow u.\texttt{leftChild}$.
   Else, we have three possibilities given in ¶15:
      If CASE (i) holds, Return(v_i). ◁ *Left tangent found:* $i_0 = i$
      If CASE (ii) holds, set $u \leftarrow u.\texttt{rightChild}$. ◁ *Go right:* $i > i_0$
      If CASE (iii) holds, set $u \leftarrow u.\texttt{leftChild}$. ◁ *Go left:* $i < i_0$
   Update i to the index of the vertex in u.

¶17. Geometric Primitives. In the above subroutines, we used "geometric primitives" such as checking whether a point is above or below a line. Such primitives must ultimately be reduced to numerical computation and comparisons. In fact, all the geometric primitives can essentially be reduced to a single primitive, the[6] **LeftTurn Predicate**. Given any three points p, q, r, define

$$\texttt{LeftTurn}(p,q,r) = \begin{cases} 0 & \text{if the points } p,q,r \text{ are collinear} \\ +1 & \text{if the path } (p,q,r) \text{ make a left turn at } q \\ -1 & \text{if the path } (p,q,r) \text{ make a right turn at } q \end{cases} \tag{8}$$

In an Exercise below, you will see how this is easily implemented as the sign of a certain 3×3 determinant. We should observe a peculiarity: we call this a "predicate" even though this is a 3-valued function. In logic, predicates are usually 2-valued (i.e., true or false). This is a general phenomenon in geometry, and we might call such 3-valued predicates a **geometric predicate** as opposed to the standard **logical predicate**.

Let us say that the input set X of points is **degenerate** if there exists three distinct points $p, q, r \in X$ such that $\texttt{LeftTurn}(p, q, r) = 0$; otherwise, X is **nondegenerate**. We will assume X is nondegenerate in the following. The main reason for this is pedagogical: the non-degenerate cases are easier to understand. We also note that there are general techniques in computational geometry for handling degeneracies, but it is beyond our scope.

[6] Also known as orientation predicate.

We now make the inner loop of the implicit search in the FINDLEFTTANGENT algorithm explicit:

```
2. Repeat:
2.1     Let u_0 = u.pred and u_1 = u.succ.   ◁ These may be NULL.
2.2     If (p <_x u)
            If (u_0 = NULL), Return(v_S).    ◁ South Pole
            else u ← u.leftChild
2.3     elif ((u_0 ≠ NULL) and LeftTurn(u_0, u, p) = 1))
            u ← u.leftChild
2.4     elif ((u_1 ≠ NULL) and LeftTurn(u, u_1, p) = −1))
            u ← u.rightChild
2.5     else Return(u).   ◁ This is correct, even if u_0 or u_1 are NULL.
```

To see why the return statement in line 2.5 is correct, first assume u_0 and u_1 are non-null. Then line 2.2 has ensured that $p <_x u$, line 2.3 has verified that (u_0, u, p) is a right-turn and line 2.4 has verified (u, u_1, p) is a left-turn. These verifications depends on non-degeneracy assumptions. The reader should next verify correctness in case u_0 or u_1 are null.

¶18. Inserting a point into an upper hull. It is now easy to implementation `insert`(p, T): we first check, using the above routine to determine if p is outside the upper hull of T. If so, we may return as `insert`(p, T) is a no-op. We first perform `tangent`(p, T) and assume the non-trivial case where a pair (q, r) of tangent points are returned. Then we need to delete from T those vertices v_i that lies strictly between q and r, and replace them by the point p. This is accomplished using the operations of `split` and `merge` on splay trees as described earlier.

We may conclude with the following. Let D be our data structure for the convex hull H. So D is a pair of splay trees representing the upper and lower hulls of H.

THEOREM 6.
(i) Using the data structure D to represent the convex hull H of a set of points, we can support `insert`(p, D) *and* `tangent`(p, D) *requests with an amortized cost of $O(\log |H|)$ time per request.*
(ii) From D, we can produce the cyclic order of points in H in time $O(|H|)$. In particular, this gives an $O(n \log n)$ algorithm for computing the convex hull of a set of n points.

This theorem gave us an $O(n \log n)$ algorithm for computing the convex of a planar point set. Under reasonable models of computation, it can be shown that $O(n \log n)$ is optimal. There are over a dozen known algorithms for convex hulls in the literature (see Exercise for some). The complexity is usually expressed as a function of n, the number of input points. But an interesting concept of "output sensitivity" is to measure the complexity in terms of n and h, where h is the size of the final convex hull. Note h is a measure of the output size, satisfying $1 \leq h \leq n$. For instance, there is a gift-wrap algorithm for convex hull that takes time $O(hn)$. So gift-wrap is faster than $O(n \log n)$ algorithm when $h = o(\log n)$. But Kirkpatrick and Seidel [4] gave an output-sensitive algorithm whose complexity is $O(n \log h)$.

Our data structure D for representing convex hulls is only semi-dynamic because we do not support the deletion of points. If we want to allow deletion of points, then points that are inside the current convex hull must be represented in the data structure. Overmars and van Leeuwen [7] designed a data structure for a fully dynamic convex hull that uses $O(\log^2 n)$ time for insertion and deletion.

There are many applications and generalizations of the convex hull problem. An obvious extension is to ask for the convex hull of a set of points in $\mathbb{R}^d$ ($d \geq 3$). Or we can replace "points" by balls or other geometric objects in $\mathbb{R}^d$. We encourage the student to explore further.

EXERCISES

Exercise 5.1: What is "convex hull" in 1-dimension? What is the upper and lower hull here? ♢

Exercise 5.2: You need some basic facts about matrices and determinants to do this problem, and to use some physical intuition. Let $a, b, c \in \mathbb{R}^2$ and consider the following 3×3 matrix,

$$M = \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{bmatrix}.$$

(i) Show that $\det M = 0$ iff a, b, c are collinear (i.e., all 3 points lies on some line). HINT: Wlog, assume $a \neq b$. Express c in terms of a and b if c lies on the line through a, b?
(ii) Show that $\det M$ is invariant if you translate all the points, i.e., (a, b, c) is replaced by $(a + t, b + t, c + t)$ for any $t \in \mathbb{R}^2$.
(iii) Show that, up to sign, $\det M$ is the twice the area of the triangle (a, b, c). HINT: from part (ii), you may assume $a = (0, 0)$ is the origin. First do the case where b, c both lie in the first quadrant, and then extend the argument, and do some simple calculations!
(iv) Show that $\det M > 0$ ($\det M < 0$) iff a, b, c list the vertices counter-clockwise (clockwise) about any point in the interior of the triangle (a, b, c).
(v) What is the relation between $\texttt{sign}(\det M))$ and $\texttt{LeftTurn}(a, b, c)$? ♢

Exercise 5.3: Prove the correctness of the FINDLEFTTANGENT(p, T) algorithm. ♢

Exercise 5.4: We want to compute the upper hull of an input sequence of points $X = (p_1, \ldots, p_n)$, where we assume that $p_1 <_x p_2 <_x \cdots <_x p_n$. We want to use an array-based approach where an upper hull $U = (v_0, \ldots, v_k)$ with $k + 1$ vertices is represented by an array $U[0..k]$, where $U[i] = v_i$. Let build U incrementally using METHOD ONE in the text, by successive insertion of p_i into the upper hull of $(p_1, \ldots, p_{i-1})$. Since the length of U can change, we must also maintain a variable to represent k. Assume the set of input points is non-degenerate (no 3 points are collinear).
(a) Suppose U currently represents the upper hull of $(p_1, \ldots, p_i)$. Describe in detail how to carry out the operation of finding LeftTangent of the next point p_{i+1} in U.
(b) Describe how to implement the insertion of a new point p_{i+1} into U.
(c) Carry out a complexity analysis of your algorithm.
(d) Show how to modify your algorithms if the input can be degenerate. You must discuss how to implement the changes using the $\texttt{LeftTurn}(p, q, r)$ predicate. NOTE: What is the correct output for U in the degenerate case? For this problem, ASSUME that an input point p is in U if it does NOT lie in the interior of the convex hull. ♢

Exercise 5.5: Treatment of Degeneracy. Recall our definition of degeneracy in the previous question.
(i) First define carefully what we mean by the convex hull (and upper hull) of X in case of degeneracies. You have two choices for this. Also define what we mean by "left-tangent" of p in case p lies on a line through two consecutive vertices of the convex hull.
(ii) Modify the FINDLEFTTANGENT(p, T) algorithm so that it works correctly for all inputs, degenerate or not. Actually, you need to describe two versions, depending on which way a degenerate convex hull is defined, etc. ♢

Exercise 5.6: Let us attend to several details in the convex hull algorithm.

(i) Show how to the non-degeneracy assumptions in the text: recall that we assume consecutive vertices on the convex hull have distinct x-coordinates, and input set X is nondegenerate.

(ii) Implement the operation `tangent`(p, H) in terms of `tangent`(p, H_U) and `tangent`(p, H_L).

(iii) Implement the operation `insert`(p, H_U).

(iv) When we inserted a new point p, we split the original tree into T_1, T_2, T_3 and then form a new splay tree rooted at p with left and right subtrees T_1, T_3. The cost of forming the new tree is $O(1)$. What is the amortized cost of this operation? ♢

Exercise 5.7: One of the simplest algorithms for convex hull is the so-called Gift-Wrapping algorithm. Start with v_1 the leftmost point of the convex hull. Now try to find v_2, v_3, etc in this order. Show that you can find the next point in $O(n)$ time. Analyze the complexity of this algorithm as a function of n and h, where $1 \leq h \leq n$ is the number of vertices on the convex hull. How does this algorithm compare to $O(n \log n)$ algorithm? ♢

Exercise 5.8: Modified Graham's algorithm for upper hulls. Let $S_n = (v_1, \ldots, v_n)$ be an input sequence of points in the plane. Assume that the points are sorted by x-coordinates and satisfy $v_1 <_x v_2 <_x \cdots <_x v_n$. (Recall "$a <_x b$" means that $a.x < b.x$.) Our goal is to compute the upper hull of S_n. In stage i ($i = 1, \ldots, n$), we have processed the sequence S_i comprising the first i points in S_n. Let H_i be the upper hull of S_i. The vertices of H_i are stored in a push-down stack data structure, D. Initially, D contain just the point v_1.

(a) Describe a subroutine $Update(v_{i+1})$ which modifies D so that it next represents the upper hull H_{i+1} upon the addition of the new point v_{i+1}. HINT: Assume D contains the sequence of points $(u_1, \ldots, u_h)$ where $h \geq 1$ and u_1 is at the top of stack, with $u_1 >_x u_2 >_x \cdots >_x u_h$. For any point p, let $LT(p)$ denote the predicate `LeftTurn`(p, u_1, u_2). If $h = 1$, $LT(p)$ is defined to be true. Implement $Update(v_{i+1})$ using the predicate $LT(p)$ and the (ordinary) operations of push and pop of D.

(b) Using part (a), describe an algorithm for computing the convex hull of a set of n points. Analyze the complexity of your algorithm.

REMARK: The amortized analysis of $S.Update(p)$ was essentially described in an Exercise (Section 1, this Chapter). Graham's original idea is to sort the vertices by their angular angle about some point p_0 in the interior of the convex hull. We must implement this with care, so as to avoid the actual computation of angles (such computation would be inexact and have robustness problems). ♢

Exercise 5.9: The divide-and-conquer for convex hull is from Shamos: divide the set into two sets X_L, X_R, each of size about $n/2$ and the two sets are separated by some vertical line L. Recursively compute their convex hulls H_L, H_R. What kind of operation(s) will allow you to compute $CH(X)$ from H_L and H_R? Show that these operations can be implemented in $O(n)$ time. ♢

END EXERCISES

§6. Fibonacci Heaps

The **Fibonacci heap data structure** invented by Fredman and Tarjan (1987) gives an efficient implementation of the mergeable queues abstract data type (ADT), which we now explain.

¶19. The mergeable queues ADT. The mergeable queues ADT involves domains of three types: *Key*, *Item* and *(mergeable)* `Queue`. As usual, each item stores a key and each queue stores a collection of items. The ADT represents a collection of queues, supporting these operations:

`makeQueue`()$\rightarrow Q$	returns an empty queue Q
`insert`(*Item* x, `Queue` Q)	
`union`(`Queue` Q_1, Q_2)	
`deleteMin`(`Queue` Q)$\rightarrow$ *Item* x	x is minimum item in Q, which is now deleted
`decreaseKey`(*Item* x, *Key* k, `Queue` Q).	

Mergeable queues are clearly extensions of priority queues (§III.2). The above operations are almost self-explanatory. In the union of Q_1, Q_2, the items in Q_2 are first moved into queue Q_1, then queue Q_2 is destroyed. Thus, the number of queues can increase or decrease over the lifetime of the data structure. The operation `deleteMin`(Q) returns a minimum item in Q, and this item is deleted from Q. This operation is unspecified in case Q is empty. In `decreaseKey`(x, k, Q), we make k the new key of x in Q. But this operation assumes k is smaller than the current key of x – otherwise, we may define it to be either an error or a null-operation (we will leave this decision unspecified).

Union is sometimes known as the **meld** operation. There may be useful operations that should be provided in practice but omitted above for the sake of economy: deleting an item, making a singleton queue, getting the minimum item without deleting it. These can be defined as follows:

`delete`(*Item* x, `Queue` Q)	$\equiv$	`decreaseKey`(x,$-\infty$,Q); `deleteMin`(Q).
`makeQueue`(*Item* x)$\rightarrow Q$	$\equiv$	`makeQueue`()$\rightarrow Q$; `insert`(x,Q).
`min`(`Queue` Q)$\rightarrow x$	$\equiv$	`deleteMin`(Q)$\rightarrow x$; `insert`(x,Q).

¶20. The Fibonacci heap data structure. Each mergeable queue is implemented by a Fibonacci heap. A Fibonacci heap H is a collection of trees $T_1, \ldots, T_m$ with these properties:

- Each tree T_i satisfies the min-heap property. In particular, the root of T_i has the minimum item in T_i.
- The roots of these trees are kept in a doubly-linked list, called the **root-list** of H.
- There are two fields H.`min`, $H.n$ associated with H. The field H.`min` points to the node with a minimum key, and $H.n$ is the number of items in H.
- For each node x in a tree T_i, we have four pointers that point to (i) the parent of x, (ii) one of its children, and (iii) two of its siblings. The sibling pointers are arranged so that all the children of x appears in a circular doubly-linked list called the **child-list** of x. If y is a child of x, the **sibling-list** of y is the child-list of x. Also, we keep track of x.`degree` (the number of children of x) and x.`mark` (a Boolean value to be explained).

This is illustrated in Figure 12. One of the trees T_0 is shown in detail: the root a of T_0 has 3 children b, c and d and they each point to a; on the other hand, a points only to b. There are two non-trivial sibling lists: (b, c, d) and (f, g).

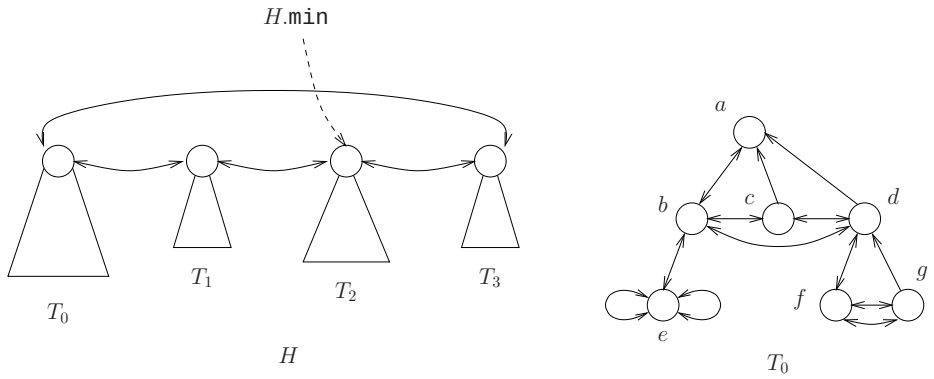


Figure 12: A Fibonacci heap $H = (T_0, \ldots, T_3)$: T_0 in detail

¶21. Linking, cutting and marking. We describe some elementary operations used in maintaining Fibonacci heaps.

(a) If x, y are two roots such that the item in x is not less than the item in y then we can **link** x **and** y; this simply makes y the parent of x. The appropriate fields and structures are then updated: x is deleted from the root-list, and then inserted into the child-list of y, the degree of y incremented, etc. This operation costs $O(1)$.

(b) The converse to linking is **cutting**. If x is a non-root in a Fibonacci heap H then we can perform *Cut*(x,H): this basically removes x from the child-list of its parent and inserts x into the root-list of H. The appropriate data variables are updated. E.g., the degree of the parent of x is decremented. Again, this operation costs $O(1)$.

(c) We say x is **marked** if $x.\texttt{mark} = true$, and **unmarked** otherwise. Initially, x is unmarked. Our rules will ensure that a root is always unmarked. We mark x if x is not a root and x loses a child (*i.e.*, a child of x is cut); we unmark x when x itself is cut (and put in the root-list). Moreover, we ensure that a marked x does not lose another child before x itself is cut (thereby reverting to unmarked status).

To do amortized analysis, we define a potential function. The **potential** of a Fibonacci heap H is defined as

$$\Phi(H) := t(H) + 2 \cdot m(H)$$

where $t(H)$ is the number of trees in H and $m(H)$ is the number marked items in H. The potential of a collection of Fibonacci heaps is just the sum of the potentials of the individual heaps.

One more definition: let $D(n)$ denote the maximum degree of a node in a Fibonacci heap with n items. We will show later that $D(n) \leq 2 \lg n$.

Remark: The reader may observe how "low-tech" this data structure appears – along with the humble array structure, linked-lists is among the simplest data structures. Yet we intend to achieve the best known overall performance for mergeable queues with Fibonacci heaps. This should be viewed as a testimony to the power of amortization.

§7. Fibonacci Heap Algorithms

We now implement the mergeable queue operations. Our goal is to achieve an amortized cost of

$O(1)$ for each operation except for `deleteMin`, which will have logarithmic amortized cost.

Recall that for each operation p, we have a **cost** $\text{COST}(p)$ which will be mostly self-evident in the following description. We must define a **charge** $\text{CHARGE}(p)$. The **credit** is thereby determined: $\text{CREDIT}(p) = \text{CHARGE}(p) - \text{COST}(p)$. This charging scheme will achieve the stated goal in the previous paragraph: $\Theta(1)$ charges for all the non-deletion operations, and $\Theta(\log n)$ for the two deletion operations. Finally, we verify the credit-potential invariant equation (2) for each operation.

`makeQueue`(): we create an empty root-list. The cost is 1, the charge is 1, so credit is 0, and finally $\Delta\Phi = 0$. The credit-potential invariant holds trivially.

> The cost and $\Delta\Phi$ is automatic at this point (our earlier decisions have determined this). Although we said that the "charge" is part of our creative design, at this point, we really have little choice if we wish to satisfy the credit-potential invariant. We might as well define charge to be (at least) the cost plus $\Delta\Phi$.

`insert`(H, x): we create a new tree T containing only x and insert T into the root-list of H. Update H.`min`, etc. Let us check the credit-potential invariant:

$$\text{COST} \leq 1, \quad \text{CHARGE} = 2, \quad \text{CREDIT} \geq 1, \quad \Delta\Phi = 1.$$

`union`(H_1, H_2): concatenate the two root-lists and call it H_1. Update $\min[H_1]$, etc. Checking the credit-potential invariant:

$$\text{COST} \leq 1, \quad \text{CHARGE} = 1, \quad \text{CREDIT} \geq 0, \quad \Delta\Phi = 0.$$

`deleteMin`(H): we remove H.`min` from the root-list, and the child-list of H.`min` can now be regarded as the root-list of another Fibonacci heap. These two circular lists can be concatenated in constant time into a new root-list for H. If t_0 is the old value of $t(H)$, the new value of $t(H)$ is at most $t_0 + D(n)$. Next we need to find the new value of H.`min`. Unfortunately, we do not know the new minimum item of H. There is no choice but to scan the new root-list of H. While scanning, we might as well[7] spend some extra effort to save future work. This is a process called **consolidation** which is explained next.

¶22. Consolidation. In this process, we are given a root-list of length L ($L \leq t_0 + D(n)$ above). We must visit every member in the root-list, and at the same time do repeated linkings *until there is at most one root of each degree*. We want to do this in $O(L)$ time. By assumption, each root has degree at most $D(n)$.

The basic method is that, for each root x, we try to find another root y of the same degree and link the two. So we create a 'new' root of degree $k+1$ from two roots of degree k. If we detect another root of degree $k+1$, we link these two to create another 'new' root of degree $k+2$, and so on. The way that we detect the presence of another root of the same degree is by indexing into an array $A[1..D(n)]$ of pointers. Initialize all entries of the array to nil. Then we scan each item x in the root-list. If $k = x$.`degree` then we try to "insert" x into $A[k]$. This means making $A[k]$ point to x. But we only do this if $A[x.\texttt{degree}] = \text{nil}$; in case $A[x.\texttt{degree}] \neq \text{nil}$, then it points to some y. In this case, link x to

[7] OK, we may be lazy but not stupid.

y or vice-versa. If x is linked to y, the latter now has degree $k+1$ and we try to "insert" y into $A[k+1]$, and so on. So each failed insertion leads to a linking, and there are at most L linking operations. Since each linking removes one root, there are at most L linkings in all. (This may not be obvious if we see this the wrong way!) Thus the total cost of consolidation is $O(L)$.

Returning to `deleteMin`, let us check its credit-potential invariant.

$$\begin{aligned} &\text{COST} \leq 1 + t_0 + D(n), \qquad \text{CHARGE} = 2 + 2D(n), \\ &\text{CREDIT} \geq 1 + D(n) - t_0, \\ &\Delta\Phi \leq 1 + D(n) - t_0. \end{aligned}$$

We need to explain our bound for $\Delta\Phi$. Let t_0, m_0 refer to the values of $t(H)$ and $m(H)$ before this `deleteMin` operation. If Φ_0, Φ_1 are (respectively) the potentials before and after this operation, then $\Phi_0 = t_0 + 2m_0$ and $\Phi_1 \leq 1 + D(n) + 2m_0$. To see this bound on Φ_1, note that no node can have degree more than $D(n)$ (by definition of $D(n)$) and hence there are at most $1 + D(n)$ trees after consolidation. Moreover, there are at most m_0 marked after consolidation. Then $\Delta\Phi = \Phi_1 - \Phi_0 \leq 1 + D(n) - t_0$, as desired.

`decreaseKey`(x, k, H): this is the remaining operation and we will exploit the marking of items in a crucial way. First, we decrease the key of x to k (first checking that $k \leq x$.`key`). If x is a root, we are done. Otherwise, let y be the parent of x. If $k \leq y$.`key`, we are done. Otherwise, we cut x. Since x is now in the root list, we need to update H.`min`, etc. If x was marked, it is now unmarked. It remains to treat y: If y is a root, we are done. Otherwise, if y was unmarked, we mark y and we are done. If y was marked (*i.e.*, has previously lost a child), then we now "recursively cut" y, using the following code fragment:

```
RECURSIVECUT(y, H):
    If (y.mark = false and y ≠ root) then y.mark:=true;
    If y ≠ root then
            Cut(y, H);
            RecursiveCut(y.parent, H).
```

Note that if $c \geq 1$ is the number of cuts, then $t(H)$ is increased by c, but $m(H)$ is decreased by $c-1$ or c (the latter iff x was marked). This implies $\Delta\Phi \leq c - 2(c-1) = 2 - c$. If

$$\text{COST} \leq c, \quad \text{CHARGE} = 2, \quad \text{CREDIT} \geq 2 - c,$$

then the credit-potential invariant is verified.

SUMMARY: we have achieved our goal of charging $O(1)$ units to every operation except for `deleteMin` which is charged $O(1) + D(n)$. We next turn to bounding $D(n)$. We remark on an unusual feature in our marking scheme: in general, each node y can suffer the loss of at most one child before y itself is made a root. But if y is already a root, we allow it to lose an unlimited number of children.

§8. Degree Bound

Our goal is to show that $D(n) = O(\log n)$.

Recall the ith Fibonacci number $i = 0, 1, 2, \ldots$ is defined by $F_i = i$ if $i = 0, 1$ and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Thus the sequence of Fibonacci numbers starts out as

$$0, 1, 1, 2, 3, 5, 8, \ldots.$$

We will use two simple facts:

(a) $F_i = 1 + \sum_{j=1}^{i-2} F_j$ for $i \geq 2$.

(b) $F_{j+2} \geq \phi^j$ for $j \geq 0$, where $\phi = (1 + \sqrt{5})/2 > 1.618$.

Fact (a) follows easily by induction, or better still, by "unrolling" the recurrence for F_i. For fact (b), we observe that ϕ is a solution to the equation $x^2 - x - 1 = 0$ so $\phi^2 = 1 + \phi$. Clearly $F_2 = 1 \geq \phi^0$ and $F_3 = 2 \geq \phi^1$. Inductively,

$$F_{j+2} = F_{j+1} + F_j \geq \phi^{j-1} + \phi^{j-2} = \phi^{j-2}(\phi + 1) = \phi^j.$$

Let x be a node in a Fibonacci heap with n items, and let

$$y_1, y_2, \ldots, y_d \tag{9}$$

be the children of x, given in the order in which they are linked to x. So $x.\texttt{degree} = d$ and y_1 is the earliest child (among $y_1, \ldots, y_d$) to be linked to x.

LEMMA 7.

$$y_i.\texttt{degree} \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i \geq 2 \end{cases}$$

Proof. This is clearly true for $i = 1$. For $i \geq 2$, note that when y_i was linked to x, the degree of x is at least $i - 1$ (since at least $y_1, \ldots, y_{i-1}$ are children of x at the moment of linking). Hence, the degree of y_i at that moment is at least $i - 1$. This is because we only do linking during consolidation, and we link two roots only when they have the same degree. But we allow y_i to lose at most one child before cutting y_i. Since y_i is not (yet) cut from x, the degree of y_i is at least $i - 2$. **Q.E.D.**

LEMMA 8. *Let* SIZE(x) *denote the number of nodes in the subtree rooted at* x *and* d *the degree of* x. *Then*

$$\text{SIZE}(x) \geq F_{2+d}, \quad d \geq 0.$$

Proof. This is seen by induction on SIZE(x). The result is true when SIZE$(x) = 1, 2$ since in these cases $d = 0, 1$, respectively. If SIZE$(x) \geq 3$, let $y_1, \ldots, y_d$ be the children of x as in (9). Then

$$\begin{aligned} \text{SIZE}(x) &= 1 + \sum_{i=1}^{d} \text{SIZE}(y_i) \\ &\geq 1 + \sum_{i=1}^{d} F_{y_i.\texttt{degree}+2} \text{ (by induction)} \\ &\geq 2 + \sum_{i=2}^{d} F_i \text{ (by last lemma)} \\ &= 1 + \sum_{i=1}^{d} F_i = F_{d+2}. \end{aligned}$$

Q.E.D.

It follows that if x has degree d, then

$$n \geq \text{SIZE}(x) \geq F_{d+2} \geq \phi^d.$$

Taking logarithms, we immediately obtain:

LEMMA 9.

$$D(n) \leq \log_\phi(n).$$

This completes our analysis of Fibonacci heaps. It is now clear why the name "Fibonacci" arises.

Background: Prior to Fibonacci heaps, the binomial heaps from Vuillemin (1978) were considered the best data structure for the mergeable queue ADT. The exercises below explores some basic properties of binomial heaps. There is some interest to improve the amortized complexity of Fibonacci heaps to worst case complexity bounds. This was finally achieved by Brodal in 2005.

EXERCISES

Exercise 8.1: Suppose that instead of cutting a node just as it is about to lose a second child, we cut a node just as it is about to lose a third child. Carry out the analysis as before. Discuss the pros and cons of this variant Fibonacci heap. ♢

Exercise 8.2:

(a) Determine $\widehat{\phi}$, the other root of the equation $x^2 - x - 1 = 0$. Numerically compute $\widehat{\phi}$ to 3 decimal places.

(b) Determine F_i exactly in terms of ϕ and $\widehat{\phi}$ HINT: $F_i = A\phi^i + B\widehat{phi}^i$ for constants A, B.

(b) What is the influence of the $\widehat{\phi}$-term on the relative magnitude of F_i? ♢

Exercise 8.3: A binomial tree is a tree whose shape is taken from $\{B_0, B_1, B_2, \ldots\}$ of trees, inductively defined as follows: the singleton node is a binomial tree denoted B_0. If x, y are roots of two binomial B_i are

... ♢

END EXERCISES

§9. Pointer Model of Computation

There is an esthetically displeasing feature in our consolidation algorithm, namely, its use of array indexing does not seem to conform to the style used in the other operations. Intuitively the reason is that, unlike the other operations, indexing does not fit within the "pointer model" of computation. In this section, we will give the pointer-based solution to consolidation. We will also look at an elegant formalization of the pointer model. This model can be used for a formal theory of computability and complexity. It has many advantages over the standard Turing machines model.

¶23. Pointer based consolidation. We outline a purely pointer-based method for consolidation. We rely on the reader's understanding of pointers as found in conventional programming languages such as `C` or `C++`.

Assume that if $k \leq D(n)$ is the maximum degree of any node (past or present) in the Fibonacci heap, we have a doubly-linked list of nodes

$$(R_0, R_1, \ldots, R_k).$$

We call this the "degree register" because every node in the heap of degree i will have a pointer to R_i. Here k is the largest degree of a node that has been seen so far. Note that when we link x to y then the degree of y increments by one and when we cut x, then the parent of x decrements by one, and these are the only possibilities. If item x has its degree changed from i to $i \pm 1$ then we can re-register x by pointing it to $R_{i\pm1}$ in constant time. Occasionally, we have to extend the length of the register by appending a new node R_{k+1} to the doubly-linked list (when some node attains a degree $k+1$ that is larger than any seen so far). It is thus easy to maintain this degree register.

Now suppose we must consolidate a root list J. By going through the items in J, we can create (with the help of the degree register) a list of lists

$$(L_0, L_1, \ldots, L_k)$$

where list L_i comprises the roots of degree i in J. This takes $O(D(n) + t)$ operations if J has t elements. It is now easy to consolidate the lists $L_0, \ldots, L_k$ into one list in which no two trees have the same degree, using $O(t)$ time. The cost of this procedure is $O(D(n) + t)$, as in the solution that uses array indexing.

But we can take this idea further: we can reinterpret the circular list H as a degree register. Whenever we register a root of degree k, we check if there is already another such root. If so, we simply link them together and recursively register the new root of degree $k+1$. The worst case cost of registration in $\Theta(\lg n)$, but the amortized cost is only $O(1)$ using the Counter Example analysis.

We get several benefits in this approach: (i) The space for the register H is $D(n) = O(\log n)$. (ii) The operation `deleteMin`(H) amounts to a simple search through this register; hence its worst case time is no longer $O(n)$ but $O(\log n)$. (iii) We have eliminated the explicit consolidation process.

¶24. The Pointer Computational Model. We now give a formal model of the pointer model. A **pointer program** Π consists of a finite sequence of instructions that operate on an implicit potentially infinite digraph G. All program variables in Π are of type POINTER, but we also manipulate integer values via these pointers. Each pointer points to some node in G. Each node N in G has four components:

(integer-value, 0-pointer, 1-pointer, 2-pointer).

These are accessed as P.`Val`, $P.0$, $P.1$ and $P.2$ where P is any pointer variable that points to N. There is a special node $N_0 \in G$ and this is pointed to by the nil pointer. By definition. nil.`Val` $= 0$ and nil.i = nil for $i = 0, 1, 2$. Note that with 3 pointers, it is easy to model binary trees.

¶25. Pointer Expressions. In general, we can specify a node by a **pointer expression**, ⟨`pointer-expr`⟩, which is either the constant nil, the NEW() operator, or has the form $P.w$ where

P is a pointer variable and $w \in \{0,1,2\}^*$. The string w is also called a **path**. Examples of pointer expressions:

$$\mathsf{nil}, \quad \textsc{New}(), \quad P, \quad P.0, \quad Q.1, \quad P.2, \quad Q.1202, \quad P.2120120$$

where P, Q are pointer variables. The NEW() operator (with no arguments) returns a returns a pointer to a "spanking new node" N where $N.0 = N.1 = N.2 = \mathsf{nil}$ and $N.\texttt{Val} = 1$. The only way to access a node or its components is via such pointer expressions.

The integer values stored in nodes are unbounded and one can perform the four arithmetic operations; compare two integers; and assign to an integer variable from any integer expression (see below).

We can compare two pointers for equality or inequality, and can assign to a pointer variable from another pointer variable or the constant nil or the function NEW(). Assignment to a nil pointer has no effect. Note that we are not allowed to do pointer arithmetic or to compare them for the "less than" relation.

The assignment of pointers can be explained with an example:

$$P.0121 \leftarrow Q.20002$$

If N is the node referenced by $P.012$ and N' is the node referenced by $Q.20002$, then we are setting $N.1$ to point to N'. If N is the nil node, then this assignment has no effect.

Naturally, we use the result of a comparison to decide whether or not to branch to a labeled instruction. Assume some convention for input and output. For instance, we may have two special pointers P_{in} and P_{out} that point (respectively) to the input and output of the program.

To summarize: a pointer program is a sequence of instructions (with an optional label) of the following types.

- Value Assignment: $\langle \texttt{pointer-expr}.Val \rangle \leftarrow \langle \texttt{integer-expr} \rangle$;
- Pointer Assignment: $\langle \texttt{path-expr} \rangle \leftarrow \langle \texttt{pointer-expr} \rangle$;
- Pointer Comparison: If $\langle \texttt{pointer-expr} \rangle = \langle \texttt{pointer-expr} \rangle$ then goto $\langle \texttt{label} \rangle$;
- Value Comparison: If $\langle \texttt{integer-expr} \rangle \geq 0$ then goto $\langle \texttt{label} \rangle$;
- Halt

Integer expressions denote integer values. For instance

$$(74 * P.000) - (Q.21 + P)$$

where P, Q are pointer variables. Here, $P.000, Q.21, P$ denotes the values stored at the corresponding nodes. Thus, an integer expression $\langle \texttt{integer-expr} \rangle$ is either

- Base Case: any literal integer constant (e.g., $0, 1, 74, -199$), a $\langle \texttt{pointer-expr} \rangle$ (e.g., $P.012, Q, \mathsf{nil}$); or
- Recursively:
$$(\langle \texttt{integer-expr} \rangle \langle op \rangle \langle \texttt{integer-expr} \rangle)$$

where $\langle op \rangle$ is one of the four arithmetic operations. Recall that nil.`Val` $= 0$. Some details about the semantics of the model may be left unspecified for now. For instance, if we divide by 0, the program may be assumed to halt instantly.

For a simple complexity model, we may assume each of the above operations take unit time regardless of the pointers or the size of the integers involved. Likewise, the space usage can be simplified to just counting the number of nodes used.

One could embellish it with higher level constructs such as while-loops. Or, we could impoverish it by restricting the integer values to Boolean values (to obtain a better accounting of the bit-complexity of such programs). In general, we could have pointer models in which the value of a node P.`Val` comes from any domain. For instance, to model computation over a ring R, we let P.`Val` be an element of R. We might wish to have an inverse to NEW(), to delete a node.

¶26. List reversal example. Consider a pointer program to reverse a singly-linked list of numbers (we only use 0-pointer of each node to point to the next node). Our program uses the pointer variables P, Q, R and we write $P \leftarrow Q \leftarrow R$ to mean the sequential assignments "$P \leftarrow Q;\quad Q \leftarrow R;$".

```
REVERSELIST:
Input: P_in, pointer to a linked list.
Output: P_out, pointer to the reversal of P_in.
      P ← nil; Q ← P_in;
      If Q = nil then goto E;
          R ← Q.0 ← P;
L:    If R = nil then goto E;
T:        P ← Q ← R ← Q.0 ← P;
      goto L;
E:    P_out ← Q.
```

This program is easy to grasp once the invariant preceding Line T is understood (see Figure 13 and Exercise).

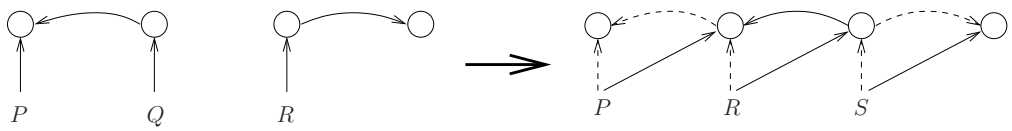


Figure 13: List Reversal Algorithm: the transformation at Line T.

Remark: This model may be more convenient than Turing machines to use as a common basis for discussing complexity theory issues. The main reservation comes from our unit cost for unbounded integers operations. In that case we can either require that all integers be bounded, or else charge a suitable cost $M(n)$ for multiplying n-bit integers, etc, reflecting the Turing machine cost. Of course, the use of pointers is still non-elementary from the viewpoint of Turing machines, but this is precisely the convenience we gain.

EXERCISES

Exercise 9.1: State the invariant before line T in the pointer reversal program; then proving the program correct. ♢

Exercise 9.2: Write the pointer program for the consolidation. ♢

Exercise 9.3: Implement in detail all the Fibonacci heap algorithms using our pointer model, ♢

Exercise 9.4: Write a sorting program and a matrix multiplication program in this model. What is the time complexity of your algorithms? ♢

END EXERCISES

§10. Application to Minimum Spanning Tree

The original application of Fibonacci heaps is in computing minimum spanning trees (MST). In Lecture IV §4, we considered Prim's algorithm for MST. The input for MST is a connected bigraph $G = (V, E; C)$ with cost function $C : E \to \mathbb{R}$.

Although our goal is to compute a minimum spanning tree, let us simplify our task by computing only the **cost** of a minimum spanning tree. This is consistent with a general point of pedagogy: for many computational problems that seek to compute a data structure $D = D^*$ which minimizes an associated cost function $f(D)$, it is easier to develop the algorithmic ideas for computing $f(D^*)$ than for computing D^*. Invariably, we could easily transform the algorithm for the minimum value $f(D^*)$ into an algorithm that produces the optimal structure D^*.

¶27. Prim-safe sets. It is easy to see that if U is a singleton then U is Prim-safe. Suppose U is Prim-safe and we ask how U might be extended to a larger Prim-safe set. Let us maintain the following information about U:
i) $\texttt{mst}[U]$, denoting the cost of the minimum spanning tree of $G|U$.
ii) For each $v \in V - U$, the least cost $\texttt{lc}_U[v]$ of an edge connecting v to U:

$$\texttt{lc}_U[v] := \min\{C(v, u) : (v, u) \in E, u \in U\}.$$

We usually omit the subscript U and just write "$\texttt{lc}[v]$" without confusion.

In order to find a node $u^* \in V - U$ with the minimum $\texttt{lc}$-value, we will maintain $V - U$ as a **single**[8] mergeable queue Q in which the least cost $\texttt{lc}[u]$ serves as the key of the node $u \in V - U$. Hence extending the Prim-safe set U by a node u^* amounts to a `deleteMin` from the mergeable queue. After the deletion, we must update the information $\texttt{mst}[U]$ and $\texttt{lc}[v]$ for each $v \in V - U$. But we do not really need to consider every $v \in V - U$: we only need to update $\texttt{lc}[v]$ for those v that are adjacent to u^*. The following code fragment captures our intent.

[8] So we are not using the full power of the mergeable queue ADT which can maintain several mergeable queues. In particular, we never perform the union operation in this application.

UPDATE(u^*, U):
1. $U \leftarrow U \cup \{u^*\}$. {This step need not be performed}
2. $\texttt{mst}[U] \leftarrow \texttt{mst}[U] + \texttt{lc}[u^*]$.
3. for v adjacent to u^* and $v \notin U$, do
   If $\texttt{lc}[v] > C[v, u^*]$ then
   $\texttt{lc}[v] \leftarrow C[v, u^*]\}$.
   DecreaseKey(v, $\texttt{lc}[v]$, Q).

We need not explicitly carry out step 1 because U is implicitly maintained as the complement of the items in Q. We now present the MST Cost version of Prim's algorithm.

MST COST ALGORITHM:
Input: $G = (V, E; C)$, a connected costed bigraph.
Output: the cost of an MST of G.
INITIALIZE:
1. $U \leftarrow \{v_0\}$; $\texttt{mst}[U] \leftarrow 0$;
2. for $v \in V - U$, do $\texttt{lc}[v] \leftarrow C(v, v_0)$;
3. Set up $V - U$ as a single mergeable queue Q:
   $Q \leftarrow MakeQueue()$;
   Insert each element of $V - U$ into Q.

LOOP:
4. while $Q \neq \emptyset$, do
   $u^* \leftarrow \texttt{deleteMin}(Q)$;
   UPDATE(u^*, U).
5. Return($\texttt{mst}[U]$).

We do not need to maintain U explicitly, although it seems clearer to put this into our pseudo-code above. In practice, the updating of U can be replaced by a step to add edges to the current MST.

¶28. Analysis. The correctness of this algorithm is immediate from the preceding discussion. To bound its complexity, let $n{:=}|V|$ and $m{:=}|E|$. Assume that the mergeable queue is implemented by a Fibonacci heap. In the UPDATE subroutine, updating the value of $\texttt{lc}[v]$ becomes a DecreaseKey operation. Each operation in UPDATE can be charged to an edge or a vertex. As each edge or vertex is charged at most once, and since the amortized cost of each operation is $O(1)$, the cost of all the updates is $O(m+n)$. The initialization takes $O(n)$ time. In the main procedure, we make $n-1$ passes through the whileloop. So we perform $n-1$ $\texttt{deleteMin}$ operations, and as the amortized cost is $O(\log n)$ per operation, this has total cost $O(n \log n)$. We have proven:

THEOREM 10. *The cost of a minimum spanning tree of a graph $(V, E; C)$ can be found in $O(|V| \log |V| + |E|)$ operations.*

¶29. Final Remarks. The amortization idea is closely related to two other topics. One is "self-organizing data structures". Originally, this kind of analysis is undertaken by assuming the input has certain probability distribution. McCabe (1965) is the first to discuss the idea of move-to-front rule. See "An account of self-organizing systems", W.J. Hendricks, *SIAM J.Comp.*, 5:4(1976); also "Heuristics that dynamically organizes data structures", James R. Bitner, *SIAM J.Comp.*, 8:1(1979)82-100. But starting from the work of Sleator and Tarjan, the competitive analysis approach has become dominant. Albers and Westbrook gives a survey in [2]. Indeed, competitive analysis is the connection to the other major topic, "online algorithms". Albers gives a survey [1].

EXERCISES

Students should be able to demonstrate understanding of Prim's algorithm by doing hand simulations. The first exercise illustrates a simple tabular form for hand simulation.

Exercise 10.1: Hand simulate Prim's algorithm on the following graph (Figure 14) beginning with v_1:

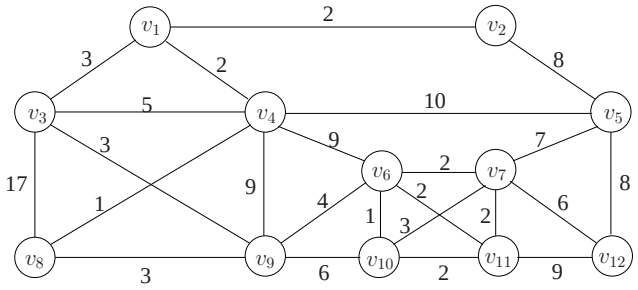


Figure 14: Graph of a House

It amounts to filling in the following table, row by row. We have filled in the first two rows already.

i	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	$mst[U]$	New Edge
1	$\underline{2}$	3	2	∞	∞	∞	∞	∞	∞	∞	∞	2	(v_1, v_2)
2	*	"	$\underline{\ }$	8	"	"	"	"	"	"	"	4	(v_1, v_4)

Note that the minimum cost in each row is underscored, indicating the item to be removed from the priority queue. ♢

Exercise 10.2: Let G_n be the graph with vertices $\{1, 2, \ldots, n\}$ and for $1 \leq i < j \leq n$, we have an edge (i, j) iff i divides j. For instance, $(1, j)$ is an edge for all $1 < j \leq n$. The **cost** of the edge (i, j) is $j - i$.

(a) Hand simulate (as in the previous exercise) Prim's algorithm on G_{10}. Show the final MST and its cost.

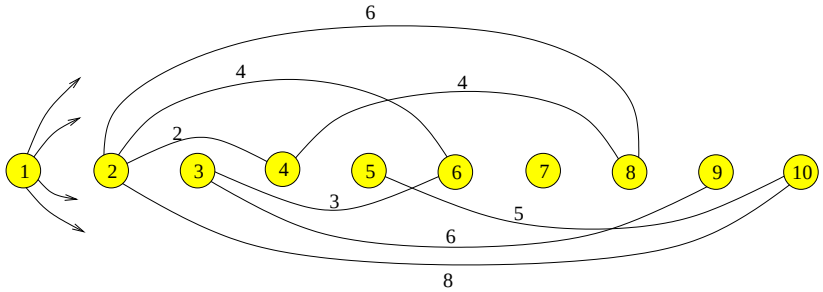


Figure 15: G_{10}: edges from node 1 are omitted for clarity.

(b) What can you say about the MST of G_n? Is it unique? What is the asymptotic cost of the MST? ♢

Exercise 10.3: Modify the above algorithm to compute a minimum spanning tree. ♢

Exercise 10.4: Modify the above algorithm to compute a minimum spanning forest in case the input graph is not connected. ♢

Exercise 10.5: Let $G = (V, E; \mu)$ be an edge-costed bigraph and $S \subseteq E$, $U \subseteq V$. Let $V(S) = \{v \in V : \exists u, (u, v) \in S\}$ denote the *vertices of* S, and $G|U{:=}(U, E'; \mu)$ where $E' = E \cap \binom{U}{2}$ denote the *restriction of* G *to* U. We define S to be *prim-safe* if S is an MST of $G|V(S)$ and S can be extended into an MST of G. We define U to be *prim-safe* if U is singleton or there exists a prim-safe set S of edges such that $U = V(S)$. Show or give a counter-example:
(a) S is a tree of $G|V(S)$ and can be extended into an MST of G implies S is prim safe.
(b) U is prim-safe implies every MST of $G|U$ is prim-safe. ♢

END EXERCISES

§11. List Update Problem

The splay tree idea originates in the "move-to-front rule" heuristic for following **list update problem**: let L be a doubly-linked list of **items** where each item has a unique key. For simplicity, we usually write L as a sequence of keys. This list supports the **access request**. Each access request r is specified by a key (also denoted r), and we satisfy this request by returning a pointer to the item in L with key r. (We assume such an item always exist.) We are interested in a special class of algorithms: such an algorithm α, on an input L and r, searches sequentially in L for the key r by starting at the head of the list. Upon finding the item with key r, α is allowed to move the item to some position nearer the head of the list (the relative ordering of the other items is unchanged). Here are three alternative rules which specify the new position of an updated item:

- (R_0) The **lazy rule** never modifies the list L.
- (R_1) The **move-to-front** rule always make updated item the new head of the list L.
- (R_2) The **transpose rule** just moves the updated item one position closer to the head of the list.

Let α_i denote the list update algorithm based on Rule R_i ($i = 0, 1, 2$). For instance, α_1 is the "move-to-front algorithm". For any algorithm α, let $COST_\alpha(r, L)$ denote the cost of an update request r on a list L using α. For $i = 0, 1, 2$, we write $COST_i(r, L)$ instead of $COST_{\alpha_i}(r, L)$. We may define $COST_i(r, L)$ to be $1 + j$ where j is the position of the accessed item in L. If α is an update algorithm, then $\alpha(L, r)$ denotes the updated list upon applying α to L, r. We extend this notation to a sequence $U = \langle r_1, r_2, \ldots, r_n \rangle$ of requests, by defining

$$\alpha(L, U) := \alpha(\alpha(L, \langle r_1, \ldots, r_{n-1} \rangle), r_n).$$

Similarly, $COST_\alpha(L, U)$ or $COST_i(L, U)$ denotes the sum of the individual update costs.

¶30. Example: Let $L = \langle a, b, c, d, e \rangle$ be a list and c an update request. Then $\alpha_0(L, c) = L$, $\alpha_1(L, c) = \langle c, a, b, d, e \rangle$ and $\alpha_2(L, c) = \langle a, c, b, d, e \rangle$. Also $COST_i(L, c) = 4$ for all $i = 0, 1, 2$.

¶31. Probabilistic Model. We analyze the cost of a sequence of updates under the lazy rule and the move-to-front rule. We first analyze a probabilistic model where the probability of updating a key k_i is p_i, for $i = 1, \ldots, m$. The lazy rule is easy to analyze: if the list is $L = \langle k_1, \ldots, k_m \rangle$ then the expected cost of a single access request is

$$C(p_1, \ldots, p_m) = \sum_{i=1}^{m} i \cdot p_i.$$

It is easy to see that this cost is minimized if the list L is rearranged so that $p_1 \geq p_2 \geq \cdots \geq p_m$; let C^* denote this minimized value of $C(p_1, \ldots, p_m)$.

What about the move-to-front rule? Let $p(i, j)$ be the probability that k_i is in front of k_j in list L. This is the probability that, if we look at the last time an update involved k_i or k_j, the operation involves k_i. Clearly

$$p(i, j) = \frac{p_i}{p_i + p_j}.$$

The expected cost to update k_i is

$$1 + \sum_{j=1, j\neq i}^{m} p(j, i).$$

The expected cost of an arbitrary update is

$$\begin{aligned}
\widehat{C} &:= \sum_{i=1}^{m} p_i \left[1 + \sum_{j=1, j\neq i}^{m} p(i, j) \right] \\
&= 1 + \sum_{i=1}^{m} \sum_{j\neq i}^{m} p_i \cdot p(i, j) \\
&= 1 + 2 \sum_{1 \leq j < i \leq m} \frac{p_i p_j}{p_i + p_j} \\
&= 1 + 2 \sum_{i=1}^{m} p_i \sum_{j=1}^{i-1} \cdot p(j, i) \\
&\leq 1 + 2 \sum_{i=1}^{m} p_i \cdot (i - 1) \\
&= 2C^* - 1.
\end{aligned}$$

This proves

$$\widehat{C} < 2C^*. \tag{10}$$

Thus, the move-to-front rule achieves close to optimal bound without having to know in advance the frequency distribution of requests, or to sort the lists.

¶32. Amortization Model. Let us now consider the amortized cost of a fixed sequence of updates

$$U = (r_1, r_2, \ldots, r_n) \tag{11}$$

on an initial list L_0 with m items. Clearly the worst case cost per update is $O(m)$. So, updates over the sequence U costs $O(mn)$. This worst case bound cannot be improved if we use the lazy rule. The best case for the lazy rule is $O(1)$ per update, or $O(n)$ overall.

What about the move-to-front rule? In analogy to equation (10), we show that it is never incur more than twice the cost of any update algorithm. In particular, it is never more than twice cost of an optimal

offline update algorithm α_*. Note that α_*, being offline, can determine the best position to move each element after it has been accessed *based on the entire sequence U in (11)*. If the cost of α_* is denoted $COST_*$, we prove

$$COST_1(L, U) \leq 2 \cdot COST_*(L, U). \tag{12}$$

We introduce the following potential function on lists. A pair (k, k') of keys is an **inversion** in a pair (L, L') of lists if k occurs before k' in L but k occurs after k' in L'. We will compare the list L produced by our move-to-front algorithm to the list L^* obtained from the optimal algorithm: the **potential** $\Phi(L)$ of L is defined to be the number of inversions in (L, L^*). For instance, $\Phi(L) = 5$ in Figure 16 as there is one inversion involving a, two inversions involving b (not counting that with a), two inversions involving c (not counting those with a or b) and 0 inversion involving d (not counting those with a, b, c).

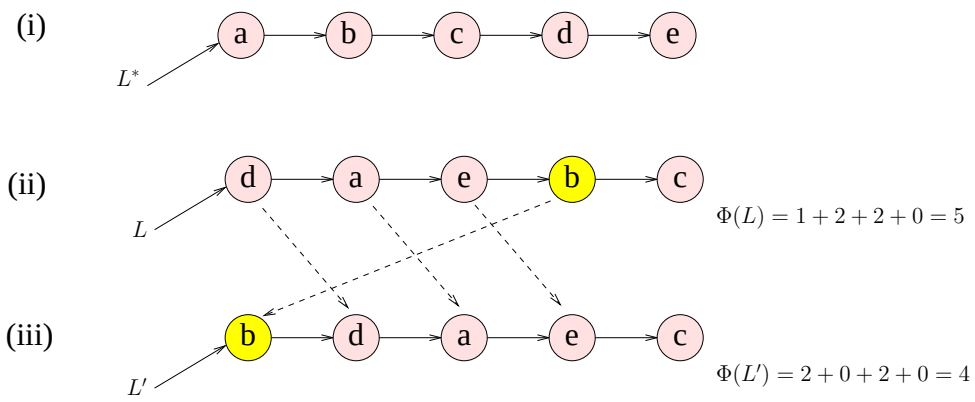


Figure 16: How potential changes under update: $L' = \alpha_1(L, b)$

Consider the jth request $(j = 1, \ldots, n)$. Let L_j (resp. L_j^*) be the list produced by the move-to-front (resp. optimal) algorithm after the jth request. Write Φ_j for $\Phi(L_j)$. Let c_j and c_j^* denote the cost of serving the jth request under two algorithms (respectively). Let x_j be the item accessed in the jth request and k_j is the number of items that are in front of x_j in both lists L_j and L_j^*. Let ℓ_j be the number of items that are in front of x_j in L_j but behind x_j in L_j^*. Hence

$$c_j = k_j + \ell_j + 1, \quad c_j^* \geq k_j + 1.$$

CLAIM: *The number of inversions destroyed is ℓ_j and the number of inversions created is at most k_j.*

In illustration, consider the list L' in Figure 16(iii), produced by the move-to-front algorithm after accessing b in L. Thus

$$x_j = b, \quad k_j = 1, \quad \ell_j = 2, \quad c_j = 4, \quad c_j^* = 2.$$

Note that ℓ_j counts the elements d and e. They represent the inversions $\{b, e\}$ and $\{b, d\}$ in L. Both inversion were **destroyed** when b moved to the front in L'. Likewise, k_j counts the element a; it represents the new inversion $\{a, b\}$ that was **created** in L' when b moved to the front. Next, the optimal algorithm α^* is allowed to update L' by moving b closer to the front of its list. Each element that b moves past when updating L' will reduce the number of created inversions by the move-to-front algorithm. It should now be clear that the above CLAIM is true.

It follows

$$\Phi_j - \Phi_{j-1} \leq k_j - \ell_j.$$

Combining these two remarks,

$$\begin{aligned} c_j + \Phi_j - \Phi_{j-1} &\leq 2k_j + 1 \\ &\leq 2c_j^* - 1. \end{aligned}$$

Summing up over all $j = 1, \ldots, n$, we obtain

$$\begin{aligned} COST_1(L_0, U) &= \left(\sum_{j=1}^{n} c_j\right) + \Phi_n - \Phi_0 \\ &\leq \sum_{j=1}^{n} (2c_j^* - 1), \qquad (\text{since } \Phi_n \geq 0, \Phi_0 = 0) \\ &= 2COST_*(L_0, U) - n. \end{aligned}$$

¶33. Competitive Algorithms. Let $\beta(k)$ be a function of k. We say an algorithm α is $\beta(k)$-**competitive** if there is some constant a, for all input lists L of length k and for all sequences U of requests

$$COST_\alpha(L, U) \leq \beta(k) \cdot COST_*(L, U).$$

Here $COST_*$ is the cost incurred by the optimal offline algorithm.

We have just shown that the Move-to-Front algorithm is 2-competitive. This idea of competitiveness from Sleator and Tarjan is an extremely powerful one as it opens up the possibility of measuring the performance of online algorithms (such as the move-to-front algorithm) without any probabilistic assumption on the input requests.

¶34. Remark. An application of the list update problem is data-compression (Exercise). Chung, Hajela and Seymour [3] determine that cost of the move-to-front rule over the cost of an optimal static ordering of the list (relative to some probability of accessing each item) is $\pi/2$. See also Lewis and Denenberg [5] and Purdom and Brown [8].

EXERCISES

Exercise 11.1: We extend the list update problem above in several ways:

(a) One way is to allow other kinds of requests. Suppose we allow insertions and deletions of items. Assume the following algorithm for insertion: we put the new item at the end of the list and perform an access to it. Here is the deletion algorithm: we access the item and then delete it. Show that the above analyses extend to a sequence of access, insert and delete requests.

(b) Extend the list update analysis to the case where the requested key k may not appear in the list.

(c) A different kind of extension is to increase the class of algorithms we analyze: after accessing an item, we allow the algorithm to to transpose any number of pairs of adjacent items, where each transposition has unit cost. Again, extend our analyses above. ♢

Exercise 11.2: The above update rules R_i ($i = 0, 1, 2$) are memoryless. The following two rules require memory.

- (R_3) The **frequency rule** maintains the list so that the more frequently accessed items occur before the less frequently accessed items. This algorithm, of course, requires that we keep a counter with each item.
- (R_4) The **timestamp rule** (Albers, 1995) says that we move the requested item x in front of the first item y in the list that precedes x and that has been requested at most once since the last request to x. If there is no such y or if x has not been requested so far, do not move x.

(a) Show that R_3 is not c-competitive for any constant c.
(b) Show that R_4 is 2-competitive. ♢

Exercise 11.3: (Bentley, Sleator, Tarjan, Wei) Consider the following data compression scheme based on any list updating algorithm. We encode an input sequence S of symbols by each symbol's position in a list L. The trick is that L is dynamic: we update L by accessing each of the symbols to be encoded. We now have a string of integers. To finally obtain a binary string as our output, we encode this string of integers by using a prefix code for each integer. In the following, assume that we use the move-to-front rule for list update. Furthermore, we use the prefix code of Elias in Exercise IV.1.1.6 that requires only

$$f(n) = 1 + \lfloor \lg n \rfloor + 2 \lfloor \lg(1 + \lg n) \rfloor$$

bits to encode an integer n.
(a) Assume the symbols are a, b, c, d, e and the initial list is $L = (a, b, c, d, e)$. Give the integer sequence corresponding to the string $S = abaabcdabaabecbaadae$. Also give the final binary string corresponding to this integer sequence.
(b) Show that if symbol x_i occurs $m_i \geq 0$ times in S then these m_i occurrences can be encoded using a total of

$$m_i f(m/m_i)$$

bits where $|S| = m$. HINT: If the positions of x_i in S are $1 \leq p_1 < p_1 < \cdots < p_{m_i} \leq m$ then the jth occurrence of x_i needs at most $f(p_j - p_{j-1})$. Then use Jensen's inequality for the concave function $f(n)$.
(c) If there are n distinct symbols $x_1, \ldots, x_n$ in S, define

$$A(S) := \sum_{i=1}^{n} \frac{m_i}{m} f\left(\frac{m}{m_i}\right).$$

Thus $A(S)$ bounds the average number of bits per symbol used by our compression scheme. Show that

$$A(S) \leq 1 + H(S) + 2 \lg(1 + H(S))$$

where

$$H(S) := \sum_{i=1}^{n} \frac{m_i}{m} \lg\left(\frac{m}{m_i}\right).$$

NOTE: $H(S)$ is the "empirical entropy" of S. It corresponds to the average number of bits per symbol achieved by the Huffman code for S. In other words, this online compression scheme achieves close to the compression of the offline Huffman coding algorithm. ♢

END EXERCISES

References

[1] S. Albers. Competitive online algorithms. BRICS Lecture Series LS-96-2, BRICS, Department of Computer Science, University of Aarhus, September 1996.

[2] S. Albers and J. Westbrook. A survey of self-organizing data structures. Research Report MPI-I-96-1-026, Max-Planck-Institut für Informatik, Im Stadtwald, D-66123 Saarbrücken, Germany, October 1996.

[3] F. R. K. Chung, D. J. Hajela, and P. D. Seymour. Self-organizing sequential search and hilbert's inequalities. *ACM Symp. Theory of Comput.*, 7, 1985. Providence, Rhode Island.

[4] D. G. Kirkpatrick and R. Seidel. The ultimate planar convex hull algorithm? *SIAM J. Comput.*, 15:287–299, 1986.

[5] H. R. Lewis and L. Denenberg. *Data Structures and their Algorithms*. Harper Collins Publishers, New York, 1991.

[6] A. Moffat, G. Eddy, and O. Petersson. Splaysort: Fast, versatile, practical. *Software - Practice and Experience*, 126(7):781–797, 1996.

[7] M. H. Overmars and J. van Leeuwen. Dynamic multi-dimensional data structures based on quad- and k-d trees. *Acta Inform.*, 17:267–285, 1982.

[8] J. Paul Walton Purdom and C. A. Brown. *The Analysis of Algorithms*. Holt, Rinehart and Winston, New York, 1985.

[9] M. Sherk. Self-adjusting k-ary search trees. In *Lecture Notes in Computer Science*, volume 382, pages 373–380, 1989. Proc. **Workshop on Algorithms and Data Structures**, Aug. 17-19, 1989, Carleton University, Ottawa, Canada.

[10] D. D. Sleator and R. E. Tarjan. Self-adjusting binary search trees. *J. ACM*, 32:652–686, 1985.

[11] R. E. Tarjan. Amortized computational complexity. *SIAM J. on Algebraic and Discrete Methods*, 6:306–318, 1985.