Lecture VII
DYNAMIC PROGRAMMING

We introduce an algorithmic paradigm called **dynamic programming**. It was popularized by Richard Bellman, circa 1954. The word "programming" here is the same term as found in "linear programming", and has the connotation of a systematic method for solving problems. The term is even identified[1] with the filling-in of entries in a table. The semantic shift from this to our contemporary understanding of the word "programming" is an indication of the progress in the field of computation.

In this chapter, we introduce dynamic programming techniques on several string problems, abstract polygon triangulation problems, and the problem of optimal binary search trees.

¶1. Divide and Conquer with a twist. Dynamic programming is a form of divide-and-conquer because it is based on solving subproblems. But it has some rather distinctive features. A simple illustration is the problem of computing the Fibonacci numbers, $F(n) = F(n-1)+F(n-2)$. On input n, the obvious recursive algorithm calls itself twice on the arguments $n-1$ and $n-2$. The returned results are added together. The running time has recurrence $T(n) = T(n-1)+T(n-2)+1$. Thus $T(n)$ is exponential (cf. §III.6, AVL trees). A little reflection shows that linear time suffices: instead of computing $F(n)$, let us define a new function $F_2(n)$ that computes a pair of consecutive Fibonacci numbers $(F(n), F(n-1))$. Then we only need one recursive call to F_2 on the argument $n-1$:

```
F_2(n):
    if (n = 1), return(1, 0)     ◁ Assume input n is ≥ 1
    (a, b) ← F_2(n − 1)          ◁ Recursive call!
    return(a + b, a)
```

The running time recurrence is now $T_2(n) = T_2(n-1)+1 = n$. Here we see the seed of the dynamic programming idea — that by keeping around solutions to several subproblems, we can avoid what would otherwise be an exponential complexity. Fibonacci, as such, is not typical because it needs only a constant number (i.e., two) subproblems. We next consider a more typical situation.

¶2. Joy Ride, again. Recall the joy ride or linear bin packing problem in Chapter V. The input a queue of riders whose weights are $w_1, \ldots, w_n$. We want to place these riders into cars, each with a weight capacity of M. We wish to place riders into cars in their queue order, while minimizing the number of cars. The new twist is that we allow negative weights (clearly our joy ride interpretation is stretched by this generalization). In any case, the greedy algorithm breaks down. For instance let $M = 5$ and $w = (4, 2, -4, 5)$. The greedy solution has two cars $(4), (2, -4, 5)$ but the optimal solution uses only one.

Ha, negative weights are children with helium balloons!

We will now give an $O(n^2)$ solution. But first, we must generalize the problem so that, instead of just solving the instance $P_n = (w_1, \ldots, w_n)$, we simultaneously solve the subproblem instances $P_i = (w_1, \ldots, w_i)$ for all $i = 1, \ldots, n$. Let b_i be the minimum number of cars for instance P_i. Now

[1] Such tables are sometimes filled out by deploying a row of human operators, each assigned to filling in some specific table entries and to pass on the partially-filled table to the next person.

the last car for instance P_n has the form $(w_i, \ldots, w_n)$ for some i and $w_i + w_{i+1} + \cdots + w_n \le M$. It is now easy to see that

$$b_n = 1 + \min\{b_i : \sum_{j=i}^{n} w_j \le M\}. \tag{1}$$

Since $b_1, \ldots, b_{n-1}$ is known, this formula shows that we can compute b_n in $O(n)$ time. We leave the correctness proof to the reader. To be explicit, we program this solution follows:

```
LINEAR BIN PACKING WITH NEGATIVE WEIGHTS:
Input: array w[1..n] containing weights and M
Output: array b[1..n] to store the values of optimal values b_i
    b[1] ← 1.
    for i = 1, ..., n
        W ← w[i]
        b[i] ← ∞
        for j = i − 1, ..., 1
            W+ = w[j]
            if (W ≤ M) b[i] ← min {b[i], 1 + b[j]}
```

The overall complexity is $T(n) = T(n-1) + n = O(n^2)$.

For instance, suppose $M = 5$ and $w = (1, 5, -2, 5, 1)$ Then $b_1 = 1$ (obviously), $b_2 = 2$, $b_3 = 2$ and $b_4 = 3$. Le us compute b_5 using the formula (**??**):

This example is typical of dynamic programming: in order to solve a problem instance P_n, we solve a polynomial number of subproblems (in this case n subproblems). In contrast, the Master recurrence handles a bounded number of subproblem instances.

¶3. From Google to Genomics. Dynamic programming is particularly effective for problems that have some underlying linear structure such strings. So many examples in this chapter involve strings. Currently, there are two major consumers of string algorithms: search engines such as Google, and computational biology. Thus, if you ask Google to search the word `strnigs`, it will ask if you meant `strings`. You can be sure that a slew of string algorithms are at work behind this innocent response. When I search for `cgtaatcc`, Google asked if I meant `ccgtcc`. It turns out that `CCGTCC.com` is the homepage for members of *Casino Chip & Gaming Token Collectors Club*. But a biologist might submit such a sequence to a database engine to find the closest match. This is because in computational genomics, a DNA sequence is just a string over the symbols `A,C,T,G`. The strings in Google and genomics have different characteristics: Google strings are words or phrases – these are much shorter than strings in biology which represent DNA or RNA sequences whose lengths go into millions. Google strings have medium size alphabets while strings in genomics have small alphabet (size 4). The corresponding algorithms should try to exploit such properties.

Let us fix some common terminology for strings. An **alphabet** is just a finite set Σ; its elements are called **letters** (or characters or symbols). A **string** (or word) is just a finite sequence of letters. The set of strings over Σ is denoted Σ^*. Let $X = x_1 x_2 \cdots x_m$ be a string where $x_i \in \Sigma$. The **length** of X is m, denoted $|X|$. The **empty string** is denoted ϵ and it has length $|\epsilon| = 0$. The ith letter of X is denoted $X[i] = x_i$ ($i = 1, \ldots, m$). Concatenation of two strings X, Y is indicated by juxtaposition, XY. Thus $|XY| = |X| + |Y|$.

§1. Longest Common Subsequence

Many string problems come down to comparing two strings for similarity. We look at one such similarity measure. A **subsequence** $Z = z_1z_2\cdots z_k$ of $X = x_1,\ldots,x_m$ is a string such that for some

$$1 \le i_1 < i_2 < \cdots < i_k \le m$$

we have $Z[\ell] = X[i_\ell]$ for all $\ell = 1,\ldots,k$. For example, `ln`, `lg` and `log` are subsequences of the string `long`.

A **common subsequence** of X, Y is a string $Z = z_1z_2\cdots z_k$ that is a subsequence of both X and Y. We call Z a **longest common subsequence** if its length $|Z| = k$ is maximum among all common subsequences of X and Y. Since the longest common subsequence may not be unique, let $LCS(X,Y)$ denote the set of longest common subsequences of X, Y. Also, let $lcs(X,Y)$ denote[2] any element of $LCS(X,Y)$: so $lcs(X,Y) \in LCS(X,Y)$. Define the numerical functions $L(X,Y) := |lcs(X,Y)|$ and $\lambda(X,Y) := |LS(X,Y)|$. Note that $\lambda(X,Y) \ge 1$ since "at worst", $LCS(X,Y)$ is the singleton comprising the empty string ϵ.

For example, if

$$X = \texttt{longest}, \qquad Y = \texttt{length} \tag{2}$$

then $LCS(X,Y) = \{\texttt{lngt}\}$, $\lambda(X,Y) = 1$ and $L(X,Y) = 4$.

Of course, the ultimate in similarity under LCS measure is when $L(X,Y) = \min\{|X|,|Y|\}$. We also mention the related concept of "substring". A subsequence Z is a **substring** of X if $X = Z'ZZ''$ for some strings Z', Z''. For instance, `on` and `g` are substrings of `long` but `ln`, `lg` and `log` are not. Thus, substrings are subsequences but the converse may not hold.

¶4. Versions of LCS. There are several versions of the **longest common subsequence (LCS) problem**. Given two strings

$$X = x_1x_2\cdots x_m, \quad Y = y_1y_2\cdots y_n,$$

the problem is to compute (respectively) one of the following:

- (Length version) Compute $L(X,Y)$
  e.g., $L(\texttt{longest},\texttt{length}) = 4$.
- (Instance version) Compute $lcs(X,Y)$
  e.g., $lcs(\texttt{longest},\texttt{length}) = \texttt{lngt}$.
- (Cardinality version) Compute $\lambda(X,Y)$
  e.g., $\lambda(\texttt{longest},\texttt{length}) = 1$.
- (Set version) Compute $LCS(X,Y)$
  e.g., $LCS(\texttt{longest},\texttt{length}) = \{\texttt{lngt}\}$.

We will mainly focus on the first two versions. The last version can be exponential if members of the set $LCS(X,Y)$ are explicitly written out; we may prefer some reasonably explicit[3] representation of $LCS(X,Y)$. We will consider representations of $LCS(X,Y)$ below.

[2] So $lcs(X,Y)$ is not really a functional notation.

[3] We can interpret "reasonably explicit" to mean that we can confirm membership in $LCS(X,Y)$ in linear time, or enumerate the members of $LCS(X,Y)$ in an efficient manner. Of course, the pair (X,Y) itself is an implicit representation of $LCS(X,Y)$, but it would fail our "reasonableness" test.

A brute force solution to the length version of the LCS problem would be to list all subsequences of length ℓ (for $\ell = m, m-1, m-2, \ldots, 2, 1$) of X, and for each subsequence to check if it is also a subsequence of Y. This is an exponential algorithm since X has 2^m subsequences. But can $|LCS(X,Y)|$ be truly exponential? Here is an example: let

$$X_n = 01a01a01a\ldots = (01a)^n, \qquad Y_n = 10a10a10a\ldots = (10a)^n. \tag{3}$$

Then $|L(X_n, Y_n)| = 2n$ (Exercise). Moreover, $|LCS(X_n, Y_n)| \geq 2^n$ since in each 01-block of X_n, we have 2 choices for matching the corresponding 10-block of Y_n. This gives us 2^n distinct LCS's.

¶5. Dynamic Programming Solution for LCS. The following is a crucial observation. Let us write X' for the prefix of X obtained by dropping the last symbol of X. This notation assumes $|X| > 0$ so that $|X'| = |X| - 1$. It is easy to verify the following formula for $L(X,Y)$:

$$L(X,Y) = \begin{cases} 0 & \text{if } mn = 0 \\ 1 + L(X', Y') & \text{if } x_m = y_n \\ \max\{L(X',Y), L(X,Y')\} & \text{if } x_m \neq y_n \end{cases} \tag{4}$$

There is a subtlety in this formula when $x_m = y_n$. The "obvious" formula for this case is

$$L(X,Y) = \max\{1 + L(X',Y'), L(X',Y), L(X,Y')\}.$$

The right hand side is simplified to the form in (1) because of

$$L(X',Y) \leq 1 + L(X',Y'),$$

and also a similar inequality involving $L(X,Y')$. Formula (1) constitutes the "dynamic programming principle" for the LCS problem – it expresses the solution for inputs of size $N = |X| + |Y|$ in terms of the solution for inputs of sizes $\leq N - 1$. We will discuss the dynamic programming principle in §4.

For any string X and natural number $i \geq 0$, let X_i denote the prefix of X of length i (if $i > |X|$, let $X_i = X$). The dynamic programming principle for $L(X,Y)$ suggests the following collection of subproblem instances:

$$L(X_i, Y_j), \qquad (i = 0, \ldots, m; j = 0, \ldots, n).$$

There are $O(mn)$ such subproblems. Note that X_0 is the empty string ϵ, so that

$$LCS(X_0, Y_j) = \{\epsilon\}, \qquad L(X_0, Y_j) = 0. \tag{5}$$

There are dynamic principles for $lcs(X,Y)$ and $LCS(X,Y)$ that are analogous to (1). Here we treat $LCS(X,Y)$, leaving $lcs(X,Y)$ as an exercise.

$$LCS(X,Y) = \begin{cases} \{\epsilon\} & \text{if } mn = 0 \\ LCS(X',Y')x_m & \text{if } x_m = y_n, \quad L(X,Y) > \max\{L(X',Y), L(X,Y')\} \\ LCS(X',Y')x_m \cup LCS(X',Y) & \text{if } x_m = y_n, \quad L(X,Y) = L(X',Y) > L(X,Y') \\ LCS(X',Y')x_m \cup LCS(X,Y') & \text{if } x_m = y_n, \quad L(X,Y) = L(X,Y') > L(X',Y) \\ LCS(X',Y')x_m \cup LCS(X,Y') \cup LCS(X',Y) & \text{if } x_m = y_n, \quad L(X,Y) = L(X,Y') = L(X',Y) \\ LCS(X',Y) & \text{if } x_m \neq y_n, \quad L(X',Y) > L(X,Y') \\ LCS(X,Y') & \text{if } x_m \neq y_n \quad L(X,Y') > L(X',Y) \\ LCS(X,Y') \cup LCS(X',Y) & \text{if } x_m \neq y_n, \quad L(X,Y') = L(X',Y). \end{cases} \tag{6}$$

Simplification: The student should compare Equations (1) and (2) to see the relative simplicity of the former equation. Also the recurrence (2) tells us that the flow of control in the algorithm for $LCS(X, Y)$ is determined by the function $L(X, Y)$. In particular, we need to compute $L(X, Y)$ if we want to compute $LCS(X, Y)$. In fact, equations (1) and (2) share a common flow of control, with some refinements for $LCS(X, Y)$. Our general strategy then is to develop an algorithm for $L(X, Y)$ first. Then we indicate the necessary modifications to yield an algorithm for $LCS(X, Y)$. This modification is usually straightforward although we will see exceptions.

Here is the dynamic programming solution for $L(X, Y)$. The algorithm sets up an $(1 + m) \times (1 + n)$ matrix $L[0..m, 0..n]$ where $L[i, j]$ is to store the value $L(X_i, Y_j)$. We fill in the entries of this matrix as follows. First fill in the 0th column and 0th row with zeros, as noted in (1). Now fill in successive rows, from left to right, using equation (1) above.

In illustration, we extend[4] the example (1) to the strings $X =$ `lengthen` and $Y =$ `elongate`:

		e	l	o	n	g	a	t	e
	0	0	0	0	0	0	0	0	0
l	0								
e	0								
n	0				x				
g	0					$1 + x$			
t	0								
h	0								
e	0								u
n	0							v	$max(u, v)$

We illustrate the formula (2) in action in two entries: the entry corresponding to the 'g'-row and 'g'-column is filled with $1 + x$ where x is the entry in the previous row and column. The entry corresponding to last row and last column is $\max(u, v)$ where u and v are the two adjacent entries. [It turns out that $x = 2, u = 5, v = 4$.] The reader may verify that $L(X, Y) = 5$ and $LCS(X, Y) = \{$`lngte`, `engte`$\}$ in this example. We leave as an exercise to program this algorithm in your favorite language.

¶6. Complexity Analysis. Each entry is filled in constant time. Thus the overall time complexity is $\Theta(mn)$. The space is also $\Theta(mn)$.

¶7. Optimal Instance or Set Computation. We said that it should be relatively easy to modify the code for computing $L(X, Y)$ to compute either $lcs(X, Y)$ or some representation of $LCS(X, Y)$. We use this observation: each entry $L[i, j]$ derives its values from one of the values in $L[i-1, j], L[i, j-1], L[i-1, j-1]$. We us modify L into a digraph G which represents $LCS(X, Y)$: the nodes of G are $(i, j) \in \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$. For each (i, j), we have an edge to $(i-1, j)$ (resp., $(i, j-1)$) if $L[i, j] = L[i-1, j]$ (resp., $L[i, j] = L[i, j-1]$). We also have an edge from (i, j) to $(i-1, j-1)$ iff $x_i = y_j$. Next, we can prune G so that only vertices and edges that lie in a path from (m, n) to $(0, 0)$ are kept. Now, given any Z, we can check if $Z \in LCS(X, Y)$ in $O(m + n)$ time in the obvious manner.

[4] No pun in-tended.

¶8. Small Space Solution. The above algorithm uses $O(mn)$ space. For Google applications, this may be acceptable; in computational genomics, this is not. We note that to fill in any row, we just need the values from two rows. In fact the space for one row is all that we need: as new entries are filled in, it can overwrite the corresponding entry of the previous row. Since a row has n entries, we just need $O(n)$ space. As rows and columns are interchangeable, we can also work with columns, so $O(\min\{m, n\})$ space suffices.

¶9. Backwards Equation. We exploit another symmetry in strings. We had been developing our equations using prefixes of X and Y. We could have equally worked with suffixes. If $X^{\#}$ denote the suffix of X obtained by omitting the first letter, then the analogue of (1) is:

$$L(X,Y) = \begin{cases} 0 & \text{if } mn = 0 \\ 1 + L(X^{\#}, Y^{\#}) & \text{if } x_1 = y_1 \\ \max\{L(X^{\#}, Y), L(X, Y^{\#})\} & \text{if } x_1 \neq y_1 \end{cases} \tag{7}$$

Let X^i denote the suffix of X length i, so $|X^i| = i$. If we use the same matrix L as before, we now need to fill in the entries in reverse order as follows:

Let $L[i, j]$ denote $L(X^{m-i}, Y^{n-j}]$. Thus, we could fill in the last row and last column with 0's immediately. If we work in row order, we can next fill in row $i - 1$ using (1), assuming row i is already filled in. The final entry to be filled in, $L[0, 0]$, contains our answer $L(X, Y)$.

¶10. Recovery of Optimal Instance in Small Space. Now we address the possibility of computing $lcs(X, Y)$ in small space. Note that the small space solution for $L(X, Y)$ does not easily extend to recovery of an optimal instance $lcs(X, Y)$. The solution to this problem requires an interesting divide-and-conquer from Hirshberg (1975).

For simplicity, assume that n is a power of two. Observe that

$$L(X,Y) = L(X_{i^*}, Y_{n/2}) + L(X^{m-i^*}, Y^{n/2}) \tag{8}$$

for some $i^* = 0, \ldots, m$. Indeed,

$$L(X,Y) = \max_{i=0,\ldots,n} \left\{ L(X_i, Y_{n/2}) + L(X^{m-i}, Y^{n/2}) \right\}. \tag{9}$$

How can we compute the i^* such that (1) holds? We use the usual (forward) recurrence to compute

$$\left\{ L(X_i, Y_{n/2}) : i = 0, \ldots, m \right\}.$$

We use the backward recurrence (1) to compute

$$\left\{ L(X^{m-i}, Y^{n/2}) : i = 0, \ldots, m \right\}.$$

This takes $O(m)$ space and $O(mn)$ time. Then using (1), we can determine i as the value that maximizes the function $L(X_i, Y_{n/2}) + L(X^{m-i}, Y^{n/2})$.

Knowing the i in (1), we could divide our lcs problem recursively into two subproblems. The key observation is that (1) can be extended into an equation for the optimal instance:

$$lcs(X,Y) = \begin{cases} \epsilon & \text{if } L(X,Y) = 0, \\ Y[1] & \text{if } n = 1 \text{ and } L(X,Y) = 1, \\ lcs(X_i, Y_{n/2}); lcs(X^{m-i}, Y^{n/2}) & \text{if } n \geq 2 \text{ and } L(X,Y) = L(X_i, Y_{n/2}) + L(X^{m-i}, Y^{n/2}). \end{cases} \tag{10}$$

where ";" denotes concatenation of strings.

The space complexity of this solution is easily shown to be $O(m)$. What about the time complexity? We have

$$T(m, n) = T(i, n/2) + T(m - i, n/2) + mn.$$

It is easy to verify by induction that $T(m, n) \le 2mn$: if $n = 1$, this is true. Otherwise,

$$\begin{aligned} T(m,n) &= T(i,n/2) + T(m-i,n/2) + mn \\ &\le 2\left(i\frac{n}{2}\right) + 2\left((m-i)\frac{n}{2}\right) + mn \\ &= 2mn. \end{aligned}$$

¶11. Other Improvements. We can exploit knowledge about the alphabet. For instance, Paterson and Masek gives an algorithm with $\Theta(mn/\log(\min(m, n)))$ time when the alphabet of the strings is bounded.

Our algorithm fill in the entries of the matrix L in a bottom-up fashion. We can also fill them in a top-down fashion. Namely, we begin by trying to fill the entry $L[m, n]$. There are 2 possibilities: (i) If $x_m = y_n$, we must recursively fill in $L[m-1, n-1]$ and then use this value to fill in $L[m, n]$. (ii) Otherwise, we must recursively fill in $L[m-1, n]$ and $L[m, n-1]$ first. In general, while trying to fill in $L[i, j]$ we must first check if the entry is already filled in (why?). If so, we can return the value at once. Clearly, this approach may lead to much fewer than mn entries being looked at. We leave the details to an exercise.

¶12. Applications. Computational problems on strings has been studied since the early days of computer science. One motivation is their application in text editors. For instance, the problem of finding a pattern in a larger string is a basic task in text editors. Another interesting application is in computer virus detection. The growth of the world wide web has been accompanied by the proliferation of computer viruses. It turns out that a virus trying to infect a computer will send messages X which are similar to a string Y peculiar to that virus. By computing $L(X, Y)$, we can measure how likely is the messages X coming from a Y-virus. See Exercise below.

The advent of computational genomics in the 1990's has brought new attention to problems on strings. We need to recall that the fundamental unit of study here is the DNA, where a DNA can be regarded as a string over an alphabet of four letters: *A, C, G, T.* These corresponds to the four bases: adenine, cytosine, guanine and thymine. DNA's can be used to identify species as well as individuals. More generally, the variations across species can be used as a basis for measuring their genetic similarity. The LCS problem is one of many that has been formulated to measure similarity. We will see another formulation in the next section. Hirschberg's space-efficient LCS algorithm is from [**?**]; see [2] for recent work on space efficient dynamic programming especially for geometric problems.

Exercises

Exercise 1.1: Find the set $LCS(X, Y)$ where

$$X = 00110011, \quad Y = 10100101.$$

Show your working (the matrix) and justify your method of extracting the longest common subsequences. ♢

Exercise 1.2: Compute $L(X, Y)$ where $X =$ `lengthening` and $Y =$ `elongation`. ♢

Exercise 1.3: Compute $LCS(X, Y)$ for $X =$ `AATTCCCCGACTGCAATTCACGCACC` and $Y =$ `GGCTTTTATTCTCCCTGTAAGT`. These are parts of DNA sequences from a modern human and a Neanderthal, respectively. ♢

Exercise 1.4:
(a) Give a direct recursive algorithm for computing $L(X, Y)$ based on equation (1) and show that it takes exponential time. (In other words, equation (1) alone does not ensure efficiency of solution.)
(b) Let $lcs(X, Y)$ denote any member of $LCS(X, Y)$. Give the analogue of (2) for $lcs(X, Y)$. ♢

Exercise 1.5: (V.Sharma and Yap) Consider the example in (1).
(a) Compute $L(X_2, Y_2)$ by filling in the the usual matrix. Moreover, determine $|LCS(X_2, Y_2)|$ by counting the number of maximum paths in the matrix.
(b) Prove that $L(X_n, Y_n) = 2n$. HINT: use induction on n.
(c) We indicated that $|LCS(X_n, Y_n)| \geq 2^n$. But prove that $|LCS(X_n, Y_n)| \geq \sqrt{6}^n$ (assume n is even). HINT: $|LCS(X_2, Y_2)| = 6$.
(d) Generalize the idea of (c) to prove larger lower bounds on $|LCS(X_n, Y_n)|$. ♢

Exercise 1.6: Let $S = \{X_1, \ldots, X_k\}$ be a set of strings. A string Z such that each X_i is a subsequence of Z is called a **superstring** of S. We can consider the corresponding "shortest superstring problem" for any given S. In some sense, this is the dual of the LCS problem. Is there a dynamic programming solution for the shortest superstring problem? ♢

Exercise 1.7: Joe Quick observed that the recurrence (1) for computing $L(X, Y)$ would work just as well if we look at suffixes of X, Y (*i.e.*, by omitting prefixes). On further reflection, Joe concluded that we could double the speed of our algorithm if we work from *both ends* of our strings! That is, for $0 \leq i < j$, let $X_{i,j}$ denote the substring $x_i x_{i+1} \cdots x_{j-1} x_j$. Similarly for $Y_{k,\ell}$ where $0 \leq k < \ell$. Derive an equation corresponding to (1) and describe the corresponding algorithm. Perform an analysis of your new algorithm, to confirm and or reject the Quick Hypothesis. ♢

Exercise 1.8: What are the forbidden configurations in the matrix M? For instance, we have the following constraints: $0 \leq M[i, j] - M[i-1, j] \leq 1$ and $0 \leq M[i, j] - M[i, j-1] \leq 1$. Also, $M[i, j] = M[i-1, j] = M[i, j-1] = M[i-1, j-1]$ is impossible. Note that these constraints are based only on adjacency matrix entries. Is it possible to exactly characterize the set of all allowable configurations of M based on such adjacency constraints? ♢

Exercise 1.9:
(a) Write the code in your favorite programming language to fill the above table for $L(X, Y)$.
(b) Modify the code so that the program retrieves some member of $LCS(X, Y)$.
(c) Modify (b) so that the program also reports whether $|LCS(X, Y)| > 1$. Remember that we do not count duplicates in $LCS(X, Y)$. ♢

Exercise 1.10: Let X, Y be strings.
(a) Prove that $L(XX, Y) \leq 2L(X, Y)$.

(b) Give an example where the inequality is strict. the best possible.
(c) Prove that $L(XX, YY) \leq 3L(X, Y)$. How tight is this upper bound? ♢

Exercise 1.11: Suppose we have a parallel computer with unlimited number of processors.
(a) How many parallel steps would you need to solve the $L(X, Y)$ problem using our recurrence (1)?
(b) Give a solution to Joe Quick's idea (previous exercise) of having an algorithm that runs twice as fast on our parallel computer. Hint: work the last two symbols of each input string X, Y in one step. ♢

Exercise 1.12: Let $\lambda(X, Y)$ denote size of the set $LCS(X, Y)$ and $\lambda(m, n)$ be the maximum of $\lambda(X, Y)$ when $|X| = m, |Y| = n$. Finally let $\lambda(n) = \lambda(n, n)$.
(a) Compute $\lambda(n)$ for $n = 1, 2, 3, 4$.
(b) Give upper and lower bounds for $\lambda(n)$. ♢

Exercise 1.13: Let $LCS'(X, Y)$ be the *multiset* of all the longest common subsequences of X and Y. That is, for each longest common subsequence $Z \in LCS(X, Y)$, we say Z has multiplicity $k\ell$ where Z occurs k (resp., ℓ) times as a subsequence of X (resp., Y). Let $\lambda'(n, m)$ and $\lambda'(n)$ be defined as in the previous exercise. Re-do the previous exercise for $\lambda'(n)$. ♢

Exercise 1.14: Modify the algorithm for $L(X, Y)$ to compute the following functions:
(a) $\lambda'(X, Y)$
(b) $\lambda(X, Y)$ ♢

Exercise 1.15: Instead of the bottom-up filling of tables, let us do a recursive top-down approach. That is, we begin by trying to fill in the entry $L[m, n]$. If $x_m = y_n$, we recursively try to fill in the entries for $L[m-1, n-1]$; otherwise, recursively solve for $L[m-1, n]$ and $L[m, n-1]$. Can you quantify the improvements in this approach? ♢

Exercise 1.16: (a) Solve the problem of computing the length $L(X, Y, Z)$ of the longest common subsequence of three strings X, Y, Z.
(b) What can you say about the complexity of the further generalization to computing $L(X_1, \ldots, X_m)$ (for $m \geq 3$). ♢

Exercise 1.17: A common subsequence of X, Y is said to be **maximal** if it is not the proper subsequence of another common subsequence of X, Y. For example, *let* is a maximal subsequence of *longest* and *length*. Let $LCS^*(X, Y)$ denotes the set of maximal common subsequences of X and Y. Design an algorithm to compute $LCS^*(X, Y)$. ♢

Exercise 1.18:

(20 Points) Researchers are using LCS computation to fight computer viruses. A virus that is attacking a machine has a predictable pattern of messages it sends to the machine. We view the concatenation of all these messages that a potential virus sends as a single string. Call the first 1000 bytes than from any source (i.e., potential virus) the **signature** of that source. Let X be the signature of an unknown source and Y is the signature of a known virus. To test the source is the Y-virus, we compute $L(X, Y)$. Empirically, suppose it is shown that if

$L(X,Y) > 500$, then that our source is likely to be Y-virus.
(a) Design a practical and efficient algorithm for the decision problem $L(X,Y,k)$ which outputs "PROBABLY VIRUS" if $L(X,Y) > k$ and "PROBABLY NOT VIRUS" otherwise. Give the pseudo-code for an efficient practical algorithm. NOTE: The obvious algorithm is to use the standard algorithm to compute $L(X,Y)$ and then compare n to k. But we want you to do better than this. HINT: There are two ideas we want you to exploit – most students only think of one idea.
(b) Quantify the complexity of your algorithm, and compare its performance to the obvious algorithm (which first computes $L(X,Y)$). First do your analysis using the general complexity parameters of $m = |X|, n = |Y|$ and k, and also $\ell = L(X,Y)$. Also discuss this for the special case of $m = n = 1000$ and $k = 500$. ♢

Exercise 1.19: A **Davenport-Schinzel sequence on** n **symbols** (or, n-**sequence** for short) is a string $X = x_1, \ldots, x_\ell \in \{a_1, \ldots, a_n\}^*$ such that $x_i \neq x_{i+1}$. The **order** of X is the smallest integer k such that there does not exist a subsequence of length $k+2$ of the form

$$a_i a_j a_i a_j \cdots a_i a_j a_i \quad \text{or} \quad a_i a_j a_i a_j \cdots a_j a_i a_j$$

where a_i and a_j alternate and $a_i \neq a_j$. Define $\lambda_k(n)$ to be the maximum length of a n-sequence of order at most k.
(a) Show that $\lambda_1(n) = n$ and $\lambda_2(n) = 2n - 1$. NOTE: for an order 2 string, a symbol may n times.
(b) Suppose X is an n-sequence of order 3 in which a_n appears at most $\lambda_3(n)/n$ times. After erasing all occurrences of a_n, we may have to erase occurrences a_i $(i = 1, \ldots, n-1)$ in case two copies of a_i becomes adjacent. We erase as few of these a_i's as necessary so that the result X' is a $(n-1)$-sequence. Show that $|X| - |X'| \leq \lambda_3(n)/n + 2$.
(c) Show that $\lambda_3(n) = O(n \log n)$ by solving a recurrence for $\lambda_3(n)$ implied by (b).
(d) Give an algorithm to determine the order of an n-sequence. Bound the complexity $T(n,k)$ of your algorithm where n is the length input sequence and $k \leq n$ the number of symbols. ♢

Exercise 1.20: Consider the generalization of LCS in which we want to compute the LCS for any input set of strings.
(a) If the input set have bounded size, give a polynomial time solution.
(b) (Maier, 1978) If the input set is unbounded, show that the problem is NP-complete. ♢

END EXERCISES

§2. Edit Distance

We now consider another problem on strings called the **edit distance problem**. The terminology comes from the general area of text editing in modern computing, but one could also give this a computational biology interpretation. Intuitively, the edit distance $D(X,Y)$ between X and Y is the minimum cost to edit X so that it turns into Y. Assuming $D(X,Y) \geq 0$, the ultimate in similarity between X and Y is captured by the relation $D(X,Y) = 0$. Thus, to find a Y in a database of strings that is the closest to X, we want to find Y to minimize $D(X,Y)$. In contrast, using the LCS measure of the last section, we would want to find Y which maximize $L(X,Y)$. We will explore some connection between $D(X,Y)$ and $L(X,Y)$ below.

As usual, we fix the alphabet Σ. For any index $i \geq 1$ and letter $a \in \Sigma$, we define the following **editing operations**

$$Ins(i, a), \quad Del(i), \quad Rep(i, a).$$

These operations, when applied to a string X, will **insert** the letter a so that it is now in position i, **delete** the ith letter, and **replace** the ith letter by a (respectively). Let

$$Ins(i, a, X), \quad Del(i, X), \quad Rep(i, a, X) \tag{11}$$

denote the respective results.

For example, suppose $X = aatcga$. Then $Ins(3, g, X) = aagtcga$, $Del(5, X) = aatca$ and $Rep(5, t, X) = aatcta$. In general, if $Y = Ins(i, a, X)$, then $|Y| = 1 + |X|$ and

$$Y[j] = \begin{cases} X[j] & \text{if} \quad j = 1, \ldots, i-1 \\ a & \text{if} \quad j = i \\ X[j+1] & \text{if} \quad j = i+1, \ldots, |X| \end{cases}$$

The other operations can be similarly characterized.

The notations in (1) are unambiguous only when i is in the "proper range". For insertion, this means $1 \leq i \leq |X| + 1$, but for deletion and replacement, this means $1 \leq i \leq |X|$. But when i is not in the proper range, we may introduce conventions for interpreting (1). The operations $Del(i)$ and $Ins(i, a)$ are inverses of each other in the following sense:

$$Del(i, Ins(i, a, X)) = X, \qquad Ins(i, b, Del(i, X)), \tag{12}$$

for some b. Whatever our conventions for handling improper indices, we would want equations such as (1) to hold.

For simplicity, however, we simply declare such operations to be undefined. In the following, we will implicitly assume that i is in the proper range whenever we apply these operations.

Let $D(X, Y)$ be the minimum number of editing operations that will transform X to Y. Clearly,

$$|(|X| - |Y|)| \leq D(X, Y) \leq \max\{|X|, |Y|\}. \tag{13}$$

The **triangular inequality** holds: for any strings X, Y, Z, it is clear that

$$D(X, Z) \leq D(X, Y) + D(Y, Z). \tag{14}$$

In fact, $D(X, Y)$ is a metric since it satisfies the usual axioms for a metric:

(i) $D(X, Y) \geq 0$ with equality iff $X = Y$.

(ii) $D(X, Y) = D(Y, X)$.

(iii) $D(X, Y)$ satisfies the triangular inequality (1).

¶13. An Infinite Edit Distance Graph. It is interesting to view the set Σ^* of all strings over a fixed alphabet Σ as vertices of an infinite bigraph $G(\Sigma)$ in which $X, Y \in \Sigma^*$ are connected by an edge iff there exists an operation of the form (1) that transforms X to Y. Paths in $G(\Sigma)$ are called **edit paths**. Thus $D(X, Y)$ is the length of the shortest path from X to Y in $G(\Sigma)$.

In analogy to (1), we have the

$$D(X,Y) = \begin{cases} \max\{|X|,|Y|\} & \text{if } mn = 0 \\ D(X',Y') & \text{if } x_m = y_n \\ 1 + \min\{D(X',Y), D(X,Y'), D(X',Y')\} & \text{if } x_m \neq y_n \end{cases} \tag{15}$$

It as a simple exercise to prove the correctness of this formula. It follows that $D(X,Y)$ can also be computed in $O(mn)$ time by the same technique of filling in entries in an $m \times n$ matrix M.

Suppose, we want to actually compute the sequence of $D(X,Y)$ edit operations that convert X to Y. Again, we expect to annotate the matrix M with some additional information to help us do this. For this purpose, let us decode equation (1) a little. There are four cases:
(a) In case $x_m = y_n$, the edit operation is a no-op.
(b) If $D(X,Y) = 1 + D(X',Y)$, the edit operation is $Del(m,X)$.
(c) If $D(X,Y) = 1 + D(X,Y')$, the edit operation is $Ins(m+1, y_n, X)$.
(d) If $D(X,Y) = 1 + D(X',Y')$, the edit operation is $Rep(m, y_n, X)$.
Hence it is enough to store two additional bits per matrix entry to reconstruct *one* possible sequence of $D(X,Y)$ edit operation.

What is the relation between $L(X,Y)$ and $D(X,Y)$? Here are some inequalities:

Lemma 1. *Let X and Y have lengths m and n. Then*

$$D(X,Y) \le m + n - 2L(X,Y).$$

and

$$D(X,Y) \ge \max\{m,n\} - L(X,Y).$$

Proof. Upper bound: if $Z \in LCS(X,Y)$ then we have $D(X,Z) \le m - L(X,Y)$ and $D(Z,Y) \le m - L(X,Y)$, Hence $D(X,Y) \le D(X,Z) + D(Z,Y) \le m + n - 2L(X,Y)$.

Lower bound: assume $m \ge n$, so it suffices to show $L(X,Y) \ge m - D(X,Y)$. Suppose we transform X to Y in a sequence of $D(X,Y)$ edit steps. But in $D(X,Y)$ steps, there is a subsequence Z of X of length $m - D(X,Y)$ that is unaffected. Hence Z is also a subsequence of Y, i.e., $L(X,Y) \ge |Z| = m - D(X,Y)$. **Q.E.D.**

These bounds are essentially the best possible: assume $m \le n$. Then for each positive $m/2 \le \ell \le m$, there are strings X,Y such that $D(X,Y) = m + n - 2\ell$ where $L(X,Y) = \ell$. For the lower bound, for each $0 \le \ell \le n$, there are strings X,Y such that $D(X,Y) = n - \ell$. See Exercises.

¶14. The Alignment Problem. Let us generalize the editing distance problem. It is motivated by computational biology where we want to "line up two DNA sequences" to compare them. Suppose X,Y are two strings. You are allowed to insert, delete and replace, just as before. What is new is the cost function. The **alignment cost function** is given by

$$\delta : (\Sigma \cup \{*\})^2 \to \mathbb{R}$$

where $*$ is a symbol not in the alphabet Σ. The interpretation is that $\delta(x,y)$ is the cost to replace x by y. If $x = *$, it means inserting y and if $y = *$ it means deleting x. The only requirement placed on δ is symmetry:

$$\delta(x,y) = \delta(y,x).$$

This requirement simplifies our discussions below and is natural in all our applications.

The **alignment distance** between strings X, Y under this cost function is denoted $A_\delta(X, Y)$, or simply $A(X, Y)$, if δ is understood. This can be modeled as the least cost path from X to Y using appropriate costs for edges of the infinite graph $G(\Sigma)$ used in edit distance.

¶15. Example. Let $\Sigma = \{\texttt{a,b,c ,..., ,x,y,z}\}$ be the letters of the English alphabet. Define

$$\delta(x,y) = \begin{cases} \delta_0 & \text{if } x = * \text{ or } y = *, \\ 0 & \text{if } x = y, \\ \delta_1 & \text{if } x, y \text{ are both consonants or both vowels}, \\ \delta_2 & \text{else.} \end{cases} \tag{16}$$

This cost function generalizes the editing distance cost in which we take into account the nature of letters that cause mismatch. For instance, with the choice

$$\delta_0 = 3, \delta_1 = 1, \delta_2 = 2, \tag{17}$$

then $A(\texttt{there,their}) = 4$ since we can replace the last two letters in the first word by their corresponding letter in the second word. This has cost 4 since using $\delta(\texttt{r,i}) = \delta(\text{e,r}) = 2$. There is no cheaper way to effect this transformation.

¶16. Example. In the original Needleman-Wunsch (1970) algorithm for alignment, it is assumed that there is a constant $\delta_0 > 0$ such that $\delta(*, y) = \delta(x, *) = \delta_0$ for all $x, y \in \Sigma$. Call δ_0 the **gap penalty**. One way to compute the alignment cost for X, Y is to first inserting $*$'s so that they result in strings X', Y' with the same length. Now we have a "direct pairing" of X', Y' and simply add up the replacement cost in a straightforward manner. To give a concrete instance, let $\Sigma = \{\text{a,g,c,t}\}$ with $\delta(x, y) = -1$ if $x = y$ and $\delta(x, y) = 1$ otherwise. Let the gap penalty $\delta_0 = 2$. If $X = \texttt{cga}$ and $Y = \texttt{acaat}$, we could first transform X to $X' = \texttt{*cga*}$ and $Y' = Y$. Then the cost is $2 - 1 + 1 - 1 + 2 = 3$. Of course, $A(X, Y)$ is the minimum over all such direct pairings of X' and Y'.

The introduction of δ is a significant generalization of the edit distance problem in two ways: first, the cost of each operation depends on the particular letters being operated upon. Second, we allow negative costs. Here are some reasons why such generalizations make sense:

1. In genomics, one might have a reason to think that the replacement of certain symbols by others are more likely and hence have a lower cost. Generally, deletions or insertions are costly.
2. In string editing, we may think of $\delta(\texttt{v}, \texttt{b})$ to be less than $\delta(\texttt{a}, \texttt{b})$ because in many keyboard layouts, it is easy confuse the keys for `v` and `b`, but less likely to confuse `a` and `b`.
3. In the standard text-editing view, it is natural to define $\delta(a, a) = 0$ for $a \in \Sigma$. But negative costs for $\delta(a, a)$ allows us to give value to positive matches, as opposed to absence of mismatches. But imagine that the FBI has a DNA bank containing the DNA sequences collected at all crime scenes. To correlate these crimes, the FBI wants to compute all pairs of DNA's in the bank whose alignment costs are minimum. We would like to ensure, for instance, that

$$(1) \qquad A(\texttt{cg}, \texttt{cgg}) > A(\texttt{cgataa}, \texttt{cggataa}), \tag{18}$$

$$(2) \qquad A(\texttt{cg}, \texttt{cc}) > A(\texttt{cgataa}, \texttt{ccataa}). \tag{19}$$

In (1), the pair $(\texttt{cg}, \texttt{cgg})$ and the pair $(\texttt{cgataa}, \texttt{cggataa})$ each requires only one deletion to achieve optimal alignment. But the second pair has many more matches, and we would like this to yield a lower alignment cost. Similarly, in (2) the pair (cg, cc) and the pair (cgataa, ccataa) each requires only a single letter replacement to achieve optimal alignment. Again, the second pair has many more matches. We can achieve the inequalities (0) if we define negative costs, $\alpha(x,x) < 0$ for all $x \in \Sigma$.

Let us give a dynamic programming solution. For simplicity, assume that there is a gap panelty $\delta_0 > 0$. With negative costs, it is possible that $A(X,Y) = \infty$ (i.e., it has arbitrarily negative cost). See Exercise. We will assume that this does not happen (e.g., if δ_0 is sufficiently large relative to other costs). For strings $X = x_1 \cdots x_m, Y = y_1 \cdots y_n$, we have the following dynamic programming principle for alignment:

$$A(X,Y) = \begin{cases} (m+n)\delta_0 & \text{if } mn = 0 \\ \min\{\delta_0 + A(X',Y), \delta_0 + A(X,Y'), \delta(x_m,y_n) + A(X',Y')\} & \text{else} \end{cases} \tag{20}$$

We can implement this using the usual procedure to fill in the entries of an $(m+1) \times (n+1)$ matrix. In the general case, we have:

$$A(X,Y) = \begin{cases} \delta(X,*) + \delta(*,Y) & \text{if } mn = 0 \\ \min\{\delta(x_m,*) + A(X',Y), \delta_(*,y_n) + A(X,Y'), \delta(x_m,y_n) + A(X',Y')\} & \text{else} \end{cases} \tag{21}$$

But see the Exercise for an alternative to (1).

The following fact is often used: suppose X^* and Y^* have the same length and are obtained by inserting $*$'s into X and Y, respectively. Suppose $X^* = x_1x_2\cdots x_t$ and $Y^* = y_1y_2\cdots y_t$. Say X^*, Y^* is a valid pair if $(x_i,y_i) \neq (*,*)$ for all i. Define $\delta^*(X^*,Y^*)$ to be $\sum_{i=1}^t \delta(x_i,y_i)$, called the **direct alignment cost** of X^*, Y^*.

Lemma 2. *$\delta(X,Y)$ is equal to the minimum of $\delta^*(X^*,Y^*)$, over valid pairs (X^*,Y^*).*

¶17. Generalizations. There are many possible generalizations of the above string problems.

- We can introduce costs associated to each type of editing operations. The implicit cost model above is the unit cost for every operation.
- The fundamental primitive in these problems is the comparison of two letters: is letter $X[i]$ equal to letter $Y[j]$ (a "match") or not (a "non-match")? We can generalize this by allowing "approximate" matching (allowing some amount of non-match) or allow generalized "patterns" (e.g., wild card letters or regular expressions).
- We can also generalize the notion of strings. Thus "multidimensional strings" is just an arrays of letters, where the array has some fixed dimension. Thus, strings are just 1-dimensional arrays. It is natural to view 2-dimensional arrays as raster images.
- Another generalization of strings is based on trees. A **string tree** is a rooted tree T in which each node v is labeled with a letter $\lambda(v)$ (from some fixed alphabet). The tree may be ordered or unordered. In a natural way, T represents a collection (order or unordered) of strings. Let P and T be two string trees. We say that P is a **(string) subtree** of T if there is 1-1 map μ from the nodes of P to the nodes of T such that
  - μ is label-preserving: $v \in P$ and $\mu(v) \in T$ has the same label.
  - μ is "parent preserving": if u is the parent of v in P then $\mu(u)$ is the parent of $\mu(v)$ in T. For ordered trees, we further insist that μ be order preserving.

In particular, if v_0 is the root of P then $\mu(P)$ is a subtree (in the usual sense of rooted trees) of T rooted a $\mu(v_0)$. We say there is a "match" at $\mu(v_0)$. Hence a basic problem is, given P and T, find a match of P in T, if any. Consider the edit distance problem for string trees. The following edit operations may be considered: (1) Relabeling a node. (2) Inserting a new child v to a node u, and making some subset of the children of u to be children of v. In the case of ordered trees, this subset must form a consecutive subsequence of the ordered children of u. (3) Deleting a child v of a node u. This is the inverse of the insertion operation. We next assign some cost γ to each of these operations, and define the edit distance $D(T, T')$ between two string trees T and T' to be the miminum cost of a sequence of operations that transforms T to T'. A natural requirement is hat $D(T, T')$ is a metric: so, $D(T, T') \geq 0$ with equality iff $T = T'$, $D(T, T') = D(T', T)$ and the triangular inequality be satisfied.

- Let $D = \{Y_1, \ldots, Y_n\}$ be a fixed set of strings, called the dictionary. Define $A(X, D) = \min\{A(X, Y_i) : i = 1, \ldots, n\}$. We would like to preprocess D so that for any given X, we can quickly compute the set of words in the dictionary that is closest to X according to the alignment distance.

Remarks: Levenshtein (1966) introduce the editing metric for strings in the context of binary codes. Needleman and Wunsch (1970), "A general method applicable to the search for similarities in the amino acid sequence of two proteins" (J.Mol.Biol., 48(3)443-53), is considered to be the first application of dynamic programming to biological sequence comparisons. Sankoff and Kruskal (1983) considered the LCS problem in computational biology applications. Applications of string tree matching problems arise in term-rewriting systems, logic programming and evolutionary biology. We refer to the collection in [1] for a state-of-the-art overview, circa 1997.

Exercises

Exercise 2.1: Compute the edit distances $D(X, Y)$ where X, Y are given:
(a) $X = 00110011$ and $Y = 10100101$.
(b) $X =$ `agacgttcgttagca` and $Y =$ `cgactgctgtatgga`. ♢

Exercise 2.2: Compute the alignment distance between $X =$ `google` and $Y =$ `yahoo` using the alignment cost (1) and (1). For this purpose, assume `y` is a consonant. Also, express $\delta(X, Y)$ as a direct alignment cost. ♢

Exercise 2.3: Suppose we compute optimal alignment $A(X, Y)$ by filling a matrix $M[0..m, 0..n]$ where $|X| = m, |Y| = n$. Let $M[i, j]$ be the optimal cost to align X_i with Y_j where X_i is the prefix of X of length i and similarly for Y_j. Assume the alignment cost function of the previous google-yahoo question. Suppose $M[i, j] = k$. What are the possible values for $M[i-1, j-1]$ as a function of k? What about $M[i-1, j+1]$ as a function of k? Justify your answer. ♢

Exercise 2.4: Compute $A(X, Y)$ where X, Y are the strings `AATTCCCGA` and `GCATATT`. Assume δ has gap penalty 2, $\delta(x, x) = -2$ and $\delta(x, y) = 1$ if $x \neq y$. You must organize this computation systematically as in the LCS problem. ♢

Exercise 2.5: Prove (1). This is an instructive exercise. ♢

Exercise 2.6: Let x, y, z be distinct letters, and $0 \le m \le n$.
(a) Prove that $D(X, Y) = m + n - 2\ell$ where $m \ge \ell \ge m/2$, $X = x^{m-\ell}z^{\ell}$ and $Y = z^{\ell}y^{n-\ell}$.
(b) Let $X = x^{m-\ell}z^{\ell}$ and $Y = y^{n-\ell}z^{\ell}$ $(0 \le \ell \le n)$ Prove that $D(X, Y) = n - \ell$. ♢

Exercise 2.7: Let X, Y be strings. Clearly, $L(XX, YY) \ge 2L(X, Y)$.
(a) Give an example where the inequality is strict.
(b) Prove that $L(XX, Y) \le 2L(X, Y)$ and this is the best possible.
(c) Prove that $L(XX, YY) \le 3L(X, Y)$.
(d) We know from (a) and (c) that $L(XX, YY) = cL(X, Y)$ where $2 \le c \le 3$. Give sharper bounds for c. ♢

Exercise 2.8: You work for Typing-R-Us, a company that produces smart word processing editors. When the user mistypes a word, you want to lookup the dictionary for the set of closest matching words.
(a) Design an alignmment cost function δ which takes into account the keyboard layout. Assuming the QWERTY layout, you would like to define $\delta(x, y)$ to be small when x, y are close to each other in this layout. Also, row distance is much smaller than column distance. Assume $\Sigma = \{$`a,b,c ,..., x,y,z`$\}$.
(b) Using your δ function, compute $A($`qwerty, quiet`$)$ and $A($`qwerty, quickly`$)$. ♢

Exercise 2.9: We explore the consequence of negative costs in the δ function. Can we have $A(X, Y) = -\infty$? Instead of (1), consider an alternative:

$$A(X, Y) = \min\{\delta_0 + A(X', Y), \delta_0 + A(X, Y'), \delta(x_m, y_n) + A(X', Y'), \delta(a, *) + A(Xa, Y), \delta(*, b) + A(X, Yb), \delta(a, b) + A(Xa, Y$$

Under what conditions is $A(X, Y)$ well-defined? ♢

Exercise 2.10: In the text, we describe one way to compute alignment distance: $A(X, Y)$ is just the minimum of $\delta(X', Y')$ where X', Y' are obtained from X, Y by inserting $*$'s so that they have the same lengths, and $\delta(X', Y')$ is defined by summing over the δ-cost of each direct pair in X', Y'. Prove that this definition agrees with the dynamic programming definition of $A(X, Y)$. ♢

Exercise 2.11: Let $D = \{Y_1, \ldots, Y_n\}$ be a fixed set of strings, called the dictionary. Let $A(X, D) = \min\{A(X, Y_i) : i = 1, \ldots, n\}$ be the minimum alignment distance between a string X and any string Y in D. How can you preprocess D so that $A(X, D)$ can be computed in faster than the obvious method? ♢

Exercise 2.12: Suppose we allow the operation of **transpose**, $\ldots ab \ldots \to \ldots ba \ldots$. Let $T(X, Y)$ be the minimum number of operations to convert X to Y, where the operations are the usual string edit operations plus transpose.
(i) Compute $T(X, Y)$ for the following inputs: $(X, Y) = (ab, c)$, $(X, Y) = (abc, c)$, $(X, Y) = (ab, ca)$ and $(X, Y) = (abc, ca)$.
(ii) Show that $T(X, Y) \ge 1 + \min\{T(X', Y), T(X, Y'), T(X', Y')\}$.
(iii) In what sense can you say that $T(X, Y)$ cannot be reduced to some simple function of $T(X', Y), T(X, Y')$ and $T(X', Y')$?
(iv) Derive a recursive formula for $T(X, Y)$. ♢

Exercise 2.13: In computational biology applications, there is interest in another kind of edit operation: namely, you are allowed to reverse a substring: if X, Y, Z are strings, then we can transform the XYZ to XY^RZ in one step where Y^R is the reverse of R. Assume that substring reversal is added to our insert, delete and replace operations. Give an efficent solution to this version of the edit distance problem. ♢

End Exercises

§3. Polygon Triangulation

We now address a different family of problems amenable to the dynamic programming approach. These problems have an abstract structure that is best explained using the notion of convex polygons.

The standard notion of a polygon P is a geometric one, and may be represented by a sequence $(v_1, \ldots, v_n)$ of **vertices** where $v_i \in \mathbb{R}^2$ is a point in the Euclidean plane. We say P is **convex** if no v_i in contained in the interior of the triangle $\Delta(v_j, v_k, v_\ell)$ formed by any other triple of points. Figure 1 shows a convex polygon with $n = 7$ vertices. An **edge** of P is a line segment $[v_i, v_{i+1}]$ between two consecutive vertices (the subscript arithmetic, "$i + 1$", is modulo n). Thus $[v_1, v_n]$ is also an edge. A **chord** is an line segment $[v_i, v_j]$ that is not an edge.

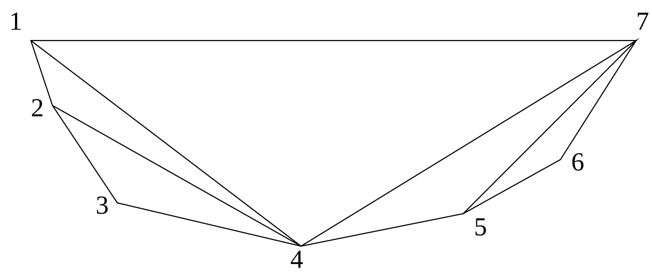


Figure 1: A triangulated 7-gon

¶18. Abstract Polygons. We now give an abstract, purely combinatorial version of these terms. Let $P = (v_1, \ldots, v_n)$, $n \geq 1$, be a sequence of n distinct symbols, called a **combinatorial convex polygon**, or an **(abstract) n-gon** for short. We call each v_i a **vertex** of P. Since the vertices are merely symbols (only the underlying linear ordering matters), it is often convenient to identify v_i with the integer i. In this case, we call $(v_1, \ldots, v_n) = (1, \ldots, n)$ the **standard n-gon**. Henceforth, we assume $n \geq 3$ to avoid trivial considerations.

Assume P is a standard n-gon. By a **segment** of P we mean an ordered pair of vertices, (i, j) where $1 \leq i < j \leq n$. This is sometimes written "ij". We classify a segment ij as an **edge** of P if $j = i + 1 (\mod n)$; otherwise the segment is called a **chord**. Thus, $1n$ is an edge. If $n \geq 3$, there are exactly n edges and $n(n - 3)/2$ chords (why?). We say two segments ij and $k\ell$ **intersect** if

$$i < k < j < \ell \qquad \text{or} \qquad k < i < \ell < j;$$

otherwise they are **disjoint**. Note that an edge is disjoint from any other segment of P.

¶19. Triangulations. It is not hard to show by induction that a *maximal* set T of pairwise disjoint chords of P has size exactly $n-3$. If $n \ge 3$, a set T with exactly $n-3$ pairwise disjoint chords is called a **triangulation** of P. In the following, it is convenient to consider the degenerate case of a 2-gon; the empty set is, by definition, the unique triangulation of a 2-gon. E.g., figure 1 shows a triangulation

$$T = \{14, 24, 47, 57\}$$

of the standard 7-gon. A **triangle** of P is a triple (i,j,k) (or simply, ijk) where $1 \le i < j < k \le n$; its three edges are ij, jk and ik. E.g., the set of all triangles of the standard 5-gon are

$$123, 124, 125, 134, 135, 145, 234, 235, 245, 345.$$

We say ijk **belongs to** a triangulation T if each edge of the triangle is either a chord in T or an edge of P. Thus the triangles of the T in figure 1 are

$$\{124, 234, 147, 457, 567\}.$$

Every triangulation T has exactly $n-2$ triangles belonging to it, and each edge of P appears as the edge of exactly one triangle and each chord in T appears as the edge of exactly two triangles [Check: $n-2$ triangles has a combined total of $2(n-3)+n$ edges.] In particular, there is a unique triangle belonging to T which contains the edge $1n$. This triangle is $(1,i,n)$ for some $i = 2, \ldots, n-1$. Then the set T can be partitioned into three disjoint subsets

$$T = T_1 \uplus T_2 \uplus S_i$$

where $S_i = T \cap \{(1,i),(i,n)\}$, and T_1, T_2 are (respectively) triangulations of the i-gon $P_1 = (1,2,\ldots,i)$ and the $(n-i+1)$-gon $P_2 = (i, i+1, \ldots, n)$. Note that $S_i = \{(1,i),(i,n)\}$ iff $2 < i < n-1$. Also, our convention about the triangulation of 2-gons is assumed when $i=2$ or $i=n-1$.

Thus triangulations can be viewed recursively. This is the key to our ability to decompose problems based on triangulations. E.g., the triangulation T in figure 1 has the partition

$$T = T_1 \uplus T_2 \uplus S_4$$

where $S_4 = \{14, 47\}$, $T_1 = \{24\}$ and $T_2 = \{57\}$.

¶20. Weight functions and optimum triangulations. A **(triangular) weight function** on n vertices is a non-negative real function W such that $W(i,j,k)$ is defined for each triangle ijk of an abstract n-gon. The W**-cost** of a triangulation T is the sum of the weights $W(i,j,k)$ of the triangles ijk belonging to T. The **optimal triangulation problem** asks for a minimum W-cost triangulation of P, given its weight function W.

▌**Example:** In case $P = (v_1, \ldots, v_n)$ is a geometric polygon in the plane, a natural cost function is $W(i,j,k)$ is the perimeter $\|v_i - v_j\| + \|v_i - v_k\| + \|v_j - v_k\|$ of the triangle (v_i, v_j, v_k), where $\|\cdot\|$ denotes the Euclidean length function. It is easy to check that T is optimal iff it minimizes the sum $\sum_{(v_i,v_j)\in T} \|v_i - v_j\|$ of the lengths of the chords in T. This might be a reasonable "cost" if a carpenter has to saw a wooden P into $n-2$ triangles. It might also be regarded as the cost of "disposing of the sawdust".

In specifying W, we generally expected the "specification size" to be $\Theta(n^3)$. However, in many applications, the function W is implicitly defined by fewer parameters, typically $\Theta(n)$ or $\Theta(n^2)$. Here are some examples.

1. **Metric Sawdust Problem:** this is a generalization of the "sawdust example". Suppose each vertex i of P is associated with a point p_i of some metric space. Then $W(i,j,k) = d(p_i,p_j) + d(p_j,p_k) + d(p_k,p_i)$ where $d(p,q)$ is the metric between two points of the space.

2. **Generalized Perimeter Problem:** W is defined by a symmetric matrix $(a_{ij})_{i,j=1}^n$ such that $W(i,j,k) = a_{ij} + a_{jk} + a_{ik}$. We can view $a_{i,j}$ as the "distance" from node i to node j and $W(i,j,k)$ is thus the perimeter of the triangle ijk. This is another generalization of "metric sawdust". Here, W is specified by $\Theta(n^2)$ parameters. More generally, we might have
$$W(i,j,k) = f(a_{ij}, a_{jk}, a_{ik})$$
where $f(\cdot,\cdot,\cdot)$ is some function.

3. **Weight functions induced by vertex weights:** W is defined by a sequence $(a_1, \ldots, a_n)$ of objects where
$$W(i,j,k) = f(a_i, a_j, a_k).$$
for some function $f(\cdot,\cdot,\cdot)$. If a_i is a number, we can view a_i as the weight of the ith vertex. Two examples are $f(x,y,z) = x + y + z$ (sum) and $f(x,y,z) = xyz$ (product). The case of product corresponds to the matrix chain product problem studied in §5.

4. **Weight functions from differences of vertex weights:** W is defined by an increasing sequence $a_1 \le a_2 \le \cdots \le a_n$ and $W(i,j,k) = a_k - a_i$. Note that the index j is not used in $W(i,j,k)$. In §5, we will see an example (optimum search trees) of such a weight function.

¶21. A dynamic programming solution. The cost of the optimal triangulation can be determined using the following recursive formula: let $C(i,j)$ be the optimal cost of triangulating the subpolygon $(i, i+1, \ldots, j)$ for $1 \le i < j \le n$. Then

$$C(i,j) = \begin{cases} 0 & \text{if } j = i+1, \\ \min_{i<k<j}\{W(i,k,j) + C(i,k) + C(k,j)\} & \text{else.} \end{cases} \qquad (22)$$

The desired optimal triangulation has cost $C(1,n)$. Assuming that the value $W(i,j,k)$ can be obtained in constant time, and the size of the input is n, it is not hard to implement this outline to give a cubic time algorithm. We say more about this in the next section.

Exercises

Exercise 3.1: Find an optimal triangulation of the abstract pentagon whose weight function W is parameterized by $(a_1, \ldots, a_6) = (4, 1, 3, 2, 2, 3)$:
(a) The weight function is given by $W(i,j,k) = a_i a_j a_k$.
(b) The weight function is given by $W(i,j,k) = |a_i - a_j| + |a_i - a_k| + |a_j - a_k|$. ♢

Exercise 3.2: Suppose P is a geometric simple polygon, not necessarily convex. We now define chords of P to comprise those segments that do not intersect the exterior of P. A triangulation is as usual a set of $n-3$ chords. Let W be a weight function on the vertices of P. Give an efficient method for computing the minimum weight triangulation of P. The goal here is to give a solution that is $O(k)$ where k is the number of chords of P. ♢

Exercise 3.3: A more profound generalization of triangulation comes from considering the triangulation (tetrahedralization) of convex polytope in 3-dimensions. Now, the number of tetrahedra is not unique. Give an abstract formulation of this problem. HINT: certain subsets of the vertices are said to be "convex". ♢

Exercise 3.4: (T. Shermer) Let P be a simple (geometric) polygon (so it need not be convex). Define the "bushiness" $b(P)$ of P to be the minimum number of degree 3 vertices in the dual graph of a triangulation of P. A triangulation is "thin" if it achieves b(P). Give an $O(n^3)$ algorithm for computing a thin triangulation. ♢

Exercise 3.5: Suppose that we want to **maximize** the "triangulation cost" (we should really interpret "cost" as "reward") for a given weight function $W(i, j, k)$. Does the same dynamic programming method solve this problem? ♢

Exercise 3.6: (Multidimensional Dynamic Programming?)
(a) Give a dynamic programming algorithm to optimally partition an n-gon into a collection of 3- or 4-gons. Assume we are given a non-negative real function $W(i, j, k, l)$, defined for all $1 \le i \le j \le k \le l \le n$ such that $|\{i, j, k, l\}| \ge 3$. The value $W(i, j, k, l)$ should depend only on the set $\{i, j, k, l\}$: if $\{i, j, k, l\} = \{i', j', k', l'\}$, then $W(i, j, k, l) = W(i', j', k', l')$. For example, $W(2, 2, 4, 7) = W(2, 4, 4, 7)$. The weight of a partitioning is equal to the sum of the weights over all 3- or 4-gons in the partition. Analyze the running time of your algorithm. NOTE: this problem has a 2-dimensional structure on its subproblems, but it can be generalized to any dimensions.
(b) Solve a variant of part (a), namely, the partition should exclusively be composed of 4-gons when $n - 4$ is even, and has exactly one 3-gon when $n - 4$ is odd. ♢

End Exercises

§4. The Dynamic Programming Method

Let us note the three ingredients necessary for a successful dynamic programming solution. We use the triangulation problem for illustration.

- **There are a small number of subproblems**. We interpret "small" to mean a polynomial number. In the weight function W on the n-gon $(1, \ldots, n)$, each contiguous subsequence

$$(i, i+1, i+2, \ldots, j-1, j), \qquad (1 \le i < j \le n)$$

  induces a weight function $W_{i,j}$ on the $(j-i+1)$-gon $(i, i+1, \ldots, j-1, j)$. This gives rise to the **subproblem** $P_{i,j}$ of optimal triangulation of $(i, i+1, \ldots, j)$. The original problem is just $P_{1,n}$. There are $\Theta(n^2)$ subproblems. The "wrong" formulation can violate this smallness requirement (see Exercise).

- **An optimal solution of a problem induces optimal solutions on certain subproblems.** If T is an optimal triangulation on $(a_1, \ldots, a_n)$, then we have noted that $T = T_1 \uplus T_2 \uplus S_i$ where $S_i \subseteq \{1i, in\}$ and T_1, T_2 are triangulations of subpolygons of P. In fact, T_1, T_2 are optimal solutions to subproblems $P_{1,i}$ and $P_{i,n}$ for some $1 < i < n$. This property is called the **dynamic programming principle**, namely, an optimal solution to a problem induces optimal solutions on certain subproblems.

- **The optimal solution of a problem is easily constructed from the optimal solutions of subproblems.** If we have already found the cost of optimal triangulations for all smaller subproblems of $P_{i,j}$ then we can easily solve $P_{i,j}$ using equation (1).

The reader may verify that the same ingredients were present in the LCS and edit distance problems.

¶22. Mechanics of the algorithm. To organize the computation embodied in equation (1), we use an upper triangular $n \times n$ matrix A to store the values of $C(i,j)$,

$$A[i,j] = C(i,j), \qquad (i < j)$$

See Figure 2.

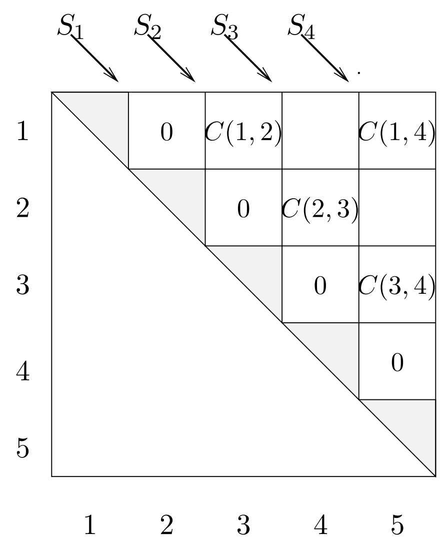


Figure 2: Filling in of a upper triangular matrix

We view the algorithm as a systematic filling in of the matrix A. Note that filling in the entries $A[i,j]$ can be viewed as solving a subproblem of size $(j-i+1)$. We proceed in $n-1$ stages, where stage S_t $(t = 2, \ldots, n)$ corresponds to solving all subproblems of size t. There are exactly $n-t+1$ problems of size t. Note that to solve a problem of size t $(t \ge 2)$ we need to minimize over a set of $t-2$ numbers (see equation (1)), and this takes time $O(t)$. Thus stage t takes $O((t-2)(n-t+1)) = O(n^2)$ time. Summed over all stages, the time is $O(n^3)$. The space requirement is $\Theta(n^2)$, because of the matrix A.

The algorithm is easy to implement in any conventional programming language: it has a triply-nested "for-loop", with the outermost loop-counter controlling the stage number, t. The following gives a bottom-up implementation of equation (1):

```
DYNAMIC PROGRAMMING FOR OPTIMAL TRIANGULATION
    for t ← 1 to n − 1    ◁ do problems of size 2
        A[t, t + 1] ← 0.
    for t ← 2 to n − 1    ◁ t + 1 is problem size
        for i ← 1 to n − t    ◁ compute C[i, i + t]
            A[i, i + t] ← A[i, i + 1] + A[i + 1, i + t] + W(i, i + 1, i + t)
            for k ← i + 2 to i + t − 1
                A[i, i + t] ← min{A[i, i + t], A[i, k] + A[k, i + t] + W(i, k, i + t)}
```

The algorithm lends itself to hand simulation, a process that the student should become familiar with.

In general, we would be filling entries of a rank k tensor (matrices are rank $k = 2$ tensors).

It is harder to visualize this process, but in terms a computer algorithm this presents no extra difficulty: we would just have a $(k+1)$-ply nested for-loop.

¶23. Splitters and the construction of Optimal Solutions. Suppose we want to find the actual optimal triangulation, not just its cost. Let us call any index k that minimizes the second expression on the right-hand side of equation (1) an **(i, j)-splitter**. If we can keep track of all the splitters, we can clearly construct the optimal triangulation. For this purpose, we employ an upper trianguler $n \times n$ matrix K where $K[i, j]$ stores an (i, j)-splitter. It is easy to see that the entry $K[i, j]$ can be filled in at the same time that $A[i, j]$ is filled in. Hence, finding optimal solutions is asymptotically the same as finding the cost of optimal solutions.

¶24. Top-down versus bottom-up dynamic programming. The above triply nested loop algorithm is a bottom-up design. However, it is not hard to construct a top-down design recursive algorithm: simply implement (1) by a recursion. However, it is important to maintain the matrices A (and K if desired) as global shared space. This technique has been called "memo-izing". Without memo-izing, the top-down solution can take exponential time, simply because there are exponentially many subproblems (see next section). A simple memoization does not speed up the algorithm. But we can, by computing bounds, avoid certain branches of the recursion. This can have potential speedup – see Exercise.

¶25. Space-Efficient Solutions. We can usually reduce the space usage by a linear factor (quadratic to linear, cubic to quadratic, etc). For instance, in the LCS problem, it is sufficient to keep at most two rows (or two columns) of the matrix in memory. That is because the solution for row i depends only on the solutions of rows $(i-1)$ and row i. Indeed, space for only one row (or column) is already sufficient – as new entries are produced for row i, they overwrite the corresponding entries or row $i-1$. However, such space efficient solutions are not so easy to extend into solutions that reconstruct the optimal solutions. For instance, how do we compute a LCS using $O(n)$ space? To do this, we need a kind of divide and conquer technique: which we explore in the exercises.

REMARK: The abstract triangulation problem has a "linear structure" on the subproblems. This linear structure can sometimes be artificially imposed on a problem in order to exploit the dynamic programming framework (see Exercise on hypercube vertex selection).

Exercises

Exercise 4.1: Jane Sharp noted an alternative to equation (1).

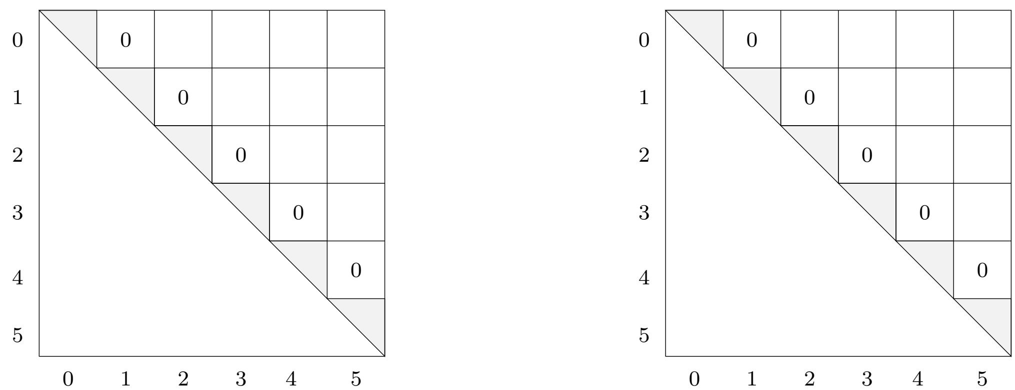
(a) Jane observed that every triangulation T must contain a triangle of the form $(i, i+1, i+2)$. Such a triangle is called an "ear". Prove this claim of Jane. (You may also prove the stronger claim that there are at least two ears.)

(b) Suppose we remove an ear from an n-gon. The result is an $(n-1)$-gon. If we knew an ear which appears in an optimum triangulation of an n-gon, we could recursively triangular the smaller $(n-1)$-gon. But since we do not know, we can try all possible $(n-1)$-gons obtained by removing an ear. What is wrong with this approach? (Try to write the analogue of equation (1), and think of the 3 ingredients needed for a dynamic programming approach.)

♢

Exercise 4.2: Consider the linear bin packing problem where the ith item is not a single weight, but a pair of non-negative weights, (v_i, w_i). If we put the ith to jth items into a bin, then we require $\sum_{k=i}^{j} v_k$ and $\sum_{k=i}^{j} w_k$ to be each bounded by M. Again the goal to use the minimum number of bins. ♢

Exercise 4.3: Let $(n_0, n_1, \ldots, n_5) = (2, 1, 4, 1, 2, 3)$. We want to multiply a sequence of matrices, $A_1 \times A_2 \times \cdots \times A_5$ where A_i is $n_{i-1} \times n_i$ for each i. Please fill in matrices (a) and (b) in Figure 3. Then write the optimal order of multiplying $A_1, \ldots, A_5$.



(a) Optimum Cost Matrix C

(b) Splitter Matrix K

Figure 3: (a) $C[i,j]$ is optimal cost to multiplying $A_i \times \cdots \times A_j$. (b) $K[i,j]$ indicates the optimal split, $(A_i \times \cdots \times A_{K[i,j]})(A_{K[i,j]+1} \times \cdots \times A_j)$

♢

Exercise 4.4: The following problem is motivated by computations in wavelet theory. We are given three real non-negative coefficients a, b, c and a real function (the "barrier")

$$h(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{else.} \end{cases}$$

Define the function $f(x, i)$ (where $i \geq 0$ is integer) as follows:

$$f(x,i) = \begin{cases} h(x) & \text{if } \ i = 0 \\ a \cdot f(2x-1, i-1) + b \cdot f(2x, i-1) + c \cdot f(2x+1, i-1) & \text{else.} \end{cases}$$

Let $f(x) = \lim_{i \to \infty} f(x, i)$. We call $f(x, i)$ the i-th approximation to $f(x)$. Assume that each arithmetic operation takes unit time.
(a) What is $f(0)$, $f(1/2)$ and $f(-1/2)$?
(b) The function $f(x, i)$ has support contained in the open interval $(-1, 1)$ (for fixed i).
(c) Prove that $f(x)$ is well-defined (possibly infinite) for all x.
(d) Determine the time to compute a single value $f(x, n)$ if we implement a straightforward recursion (each call to $f(y, i)$ is independent).
(e) We want an efficient solution for the following problem: given n, m, we want to compute the values $f(i/m, n)$ for all

$$i \in D_m := \{-m+1, -m+2, \ldots, -1, 0, 1, \ldots, m-2, m-1\}.$$

Show that this can be computed in $O(mn)$ time and $O(m)$ space.
(f) Strengthen (e) to show we can compute a single value $f(i/m, n)$ in $O(n)$ time and $O(1)$ space. ♢

Exercise 4.5: (Recursive Dynamic Programming) The "bottom-up" solution of the optimal triangulation problem is represented by a triply-nested for-loop in the text. Now we want to consider a "top-down" solution, by using recursion. As usual, the weight $W(i, j, k)$ is easily computed for any $1 \le i < j < k \le n$.
(a) Give a naive recursive algorithm for optimal triangulation. Briefly explain how this algorithm is exponential.
(b) Describe an efficient recursive algorithm. You will need to use some global data structure for sharing information across subproblems.
(c) Briefly analyze the complexity of your solution.
(d) Does your algorithm ever run faster than the bottom-up implementation? Can you make it run faster on some inputs? HINT: for subproblem $P(i, j)$, we can try to compute upper and lower bounds on $C(i, j)$. Use this to "prune" the search. ♢

Exercise 4.6: Give a linear space $O(n)$ solution to problem of optimal triangulation. Write the recurrence for the space and time complexity of your algorithm. Solve for the running time. ♢

Exercise 4.7: Consider the problem of evaluating the determinant of an $n \times n$ matrix. The obvious co-factor expansion takes $\Theta(n \cdot n!)$ arithmetic operations. Gaussian elimination takes $\Theta(n^3)$. But for small n and under certain circumstances, the co-factor method may be better. In this question, we want you to improve the co-factor expansion method by using dynamic programming. What is the number of arithmetic operations if you use dynamic programming? Please illustrate your result for $n = 3$.

HINT: We suggest you just count the number of multiplications. Then argue separately that the number of additions is of the same order. ♢

Exercise 4.8: Generalize the previous exercise. Let the set of real constants $\{a_i : i = -N, -N+1, \ldots, -1, 0, 1, \ldots, N\}$ be fixed. Suppose that

$$f(x, i) = \begin{cases} h(x) & \text{if } i = 0 \\ \sum_{i=-N}^{N} a_i \cdot f(2x - 1, i - 1) & \text{else.} \end{cases}$$

Re-do parts (a)–(c) in the last exercise. ♢

Exercise 4.9: (Hypercube vertex selection) A **hypercube** or n-**cube** is the set $H_n = \{0, 1\}^n$. Each $x = (x_1, \ldots, x_n) \in H_n$ is called a vertex of the hypercube. Let $\pi = (\pi_1, \ldots, \pi_n)$ and $\rho = (\rho_1, \ldots, \rho_n)$ be two positive integer vectors. The **price** and **reliability** of a vertex x is given by $\pi(x) = \sum_{i=1}^{n} x_i \pi_i$ and $\rho(x) = \prod_{i=1; x_i=1}^{n} \rho_i$. The **hypercube vertex selection problem** is this: given π, ρ and a positive bound B_0, find $x \in H_n$ which maximizes $\rho(x)$ subject to $\pi(x) \le B_0$. Solve this problem in time $O(nB_0)$ (not $O(n \log B_0)$).
HINT: View $H_n = H_k \otimes H_{n-k}$ for any $k = 1, \ldots, n-1$ and $y \otimes z$ denotes concatenation of vectors $y \in H_k$, $z \in H_{n-k}$. Solve subproblems on H_k and H_{n-k} with varying values of B ($B = 1, 2, \ldots, B_0$). The choice of k is arbitrary, but what is the best choice of k? ♢

Exercise 4.10: Let $S \subseteq \mathbb{R}^2$ be a set of n points. Partially order the points $p = (p.x, p.y) \in \mathbb{R}^2$ as follows: $p \le q$ iff $p.x \le q.x$ and $p.y \le q.y$. If $p \ne q$ and $p \le q$, we write $p < q$. A point p is S-**minimal** if $p \in S$ and there does not exist $q \in S$ such that $q < p$. Let $\min(S)$ denote the set of S-minimal points.
(a) For $c \in R$, let $S(c)$ denote the set $\{p \in S : p.x \ge c\}$. E.g., let $S =$

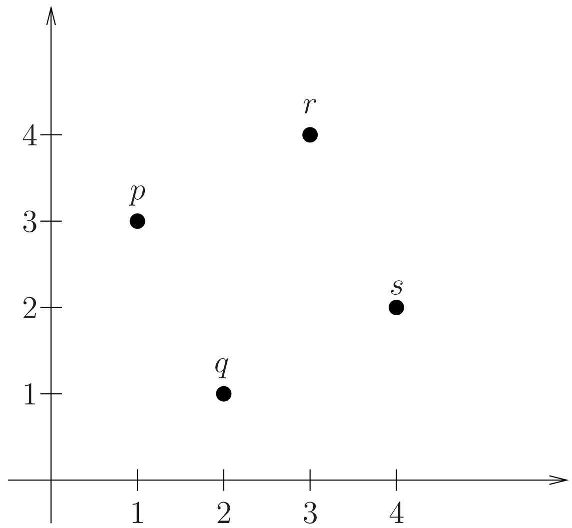


Figure 4: Set of 4 points.

$\{p(1,3), q(2,1), r(3,4), s(4,2)\}$ as shown in figure 4. Then $\min(S(c))$ is equal to $\{p,q\}$ if $c \le 1$; $\{q\}$ if $1 < c \le 2$; $\{r,s\}$ if $2 < c \le 3$; $\{s\}$ if $3 < c$. Design a data structure $D(S)$ with two properties:

1. For any $c \in \mathbb{R}$ ("the query" is specified by c), you can use $D(S)$ to output the set $\min(S(c))$ in time
$$O(\log n + k)$$
where k is the size of $\min(S(c))$.
2. The data structure $D(S)$ uses $O(n)$ space.

(b) For any $q \in \mathbb{R}^2$, let $S(q)$ denote the set $\{p \in S : p.x \ge q.x, p.y \ge q.y\}$. Design a data structure $D'(S)$ such that for any $q \in \mathbb{R}^2$, you can use $D''(S)$ to output the set $\min(S(q))$ in time $O(\log n + k)$ where k is the size of $\min(S(q))$, and $D''(S)$ uses $O(n^2)$ space. ♢

Exercise 4.11: (Knapsack) In this problem, you are given $2n+1$ positive integers,
$$W, w_i, v_i (i = 1, \ldots, n).$$
Intuitively, W is the size of your knapsack and there are n items where the ith item has size w_i and value v_i. You want to choose a subset of the items of maximum value, subject to the total size of the selected items being at most W. Precisely, you are to compute a subset $I \subseteq \{1, \ldots, n\}$ which maximizes the sum
$$\sum_{i \in I} v_i$$
subject to the constraint $\sum_{i \in I} w_i \le W$.
(a) Give a dynamic programming solution that runs in time $O(nW)$.
(b) Improve the running time to $O(n, \min\{W, 2^n\})$. ♢

Exercise 4.12: (Optimal line breaking) This book (and most technical papers today) is typeset using Donald Knuth's computer system known as TEX. This remarkable system produces very high quality output because of its sophisticated algorithms. One such algorithm is the way in which it breaks a paragraph into individual lines.

A **paragraph** can be regarded as a sequence of words. Suppose there are n words, and their lengths are $a_1, \ldots, a_n$. The problem is to break the paragraph into lines, no line having length more than m. Between 2 words in a line we introduce one space; there is no spaces after the last word in a line. If a line has length k, then we assess a **penalty** of $m - k$ on that line. The penalty for a particular method of breaking up a paragraph is the sum of the penalty over all lines. The last line of a paragraph, by definition, suffers no penalty.
(a) Consider the obvious greedy method to solve this problem (basically fill in each line until the next word will cause an overflow). Give an example to show that this does not always give the minimum penalty solution.
(b) Give a dynamic programming solution to finding the optimal (i.e., minimal penalty) solution.
(c) Illustrate your method with Lincoln's Gettysburg address, assuming that $m = 80$. In the case of a terminal word (which is followed by a full-stop), we consider the full stop as part of the word.
(d) Suppose we assume that there are 2 spaces separating a full-stop and the following word (if any) in the line. Modify your solution in (a) to handle this.
(e) Now introduce optional hyphenation into the words. For simplicity, assume that every word has zero or one potential place for hyphenation (the algorithm is told where this hyphen can be placed). If an input word of length ℓ can be broken into two half-words of lengths ℓ_1 and ℓ_2, respectively, it is assumed that $\ell_1 \geq 2$ and $\ell_2 \geq 1$. Furthermore, we must include an extra unit (for the placement of the hyphen character) in the length of the line that contains the first half. Can you modify the above algorithm further? ♢

End Exercises

§5. Optimal Parenthesization

We can view a triangulation of an $(n+1)$-gon to be a "parenthesized expression" on n symbols. Let us clarify this connection.

Let $(e_1, e_2, \ldots, e_n)$, $n \geq 1$, be a sequence of n symbols. A **(fully) parenthesized expression** on $(e_1, \ldots, e_n)$ is one whose atoms are e_i (for $i = 1, \ldots, n$), each e_i occurring exactly once and in this order left-to-right, and where each matched pair of parenthesis encloses exactly two non-empty subexpressions. E.g., there are exactly two parenthesized expressions on $(1, 2, 3)$:

$$((12)3), \qquad (1(23)).$$

The reader may verify that there are 5 parenthesized expressions on $(1, 2, 3, 4)$.

A parenthesized expression on $(e_1, \ldots, e_n)$ corresponds bijectively to a **parenthesis tree** on $(e_1, \ldots, e_n)$. Such a tree is a full[5] binary tree T on n leaves, where the ith leaf in symmetric order is associated with e_i. If $n = 1$, then the tree has only one node. Otherwise, the left and right subtrees are (respectively) parenthesized expressions on $(e_1, \ldots, e_i)$ and $(e_{i+1}, \ldots, e_n)$ for some $i = 1, \ldots, n$.

There is a slightly more involved bijective correspondence between parenthesis trees on $(e_1, \ldots, e_n)$ and triangulations of an abstract $(n + 1)$-gon. See Figure 5 for an illustration. If

[5] A node of a binary tree is **full** if it has two children. A binary tree is **full** if every internal node is full.

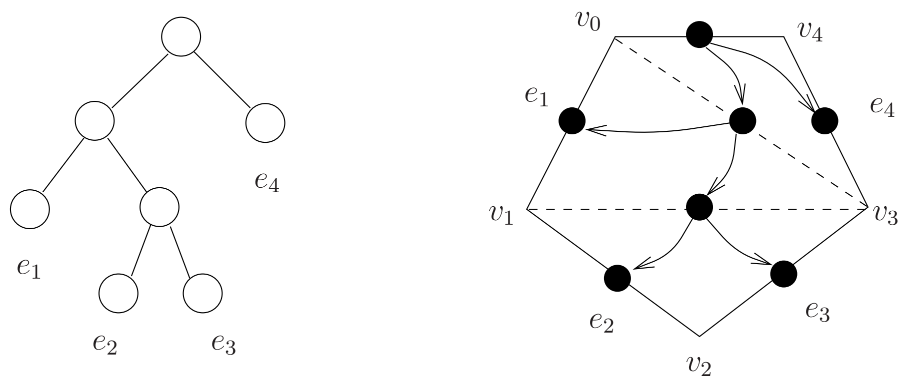


Figure 5: The parenthesis tree and triangulation corresponding to $((e_1(e_2e_3))e_4)$.

the $(n+1)$-gon is $(v_0, v_1, \ldots, v_n)$, then the edges (v_{i-1}, v_i) is mapped to e_i $(i = 1, \ldots, n)$ under this correspondence, but the "distinguished edge" (v_0, v_n) is not mapped. We leave the details for an exercise.

If we associate a cost $W(i, j, k)$ for forming a parenthesis of the form "(E_1, E_2)" where E_1 (resp., E_2) is a parenthesized expression on $(e_i, \ldots, e_j)$ (resp., $(e_{j+1}, \ldots, e_k)$, then we may speak of the **cost** of a parenthesized expression – it is the same as the cost of the corresponding triangulation of P. Finding such an optimal parenthesized expression on $(e_1, \ldots, e_n)$ is clearly equivalent to finding an optimal triangulation of P.

¶26. Catalan numbers. It is instructive to count the number $P(n)$ of parenthesis trees on $n \geq 1$ leaves. In the literature, $P(n)$ is also denoted $C(n-1)$, in which case it is called a **Catalan number**. The index $n-1$ of the Catalan numbers is the number of pairs of parenthesis needed to parenthesize n symbols. Here $C(n) = 1, 1, 2, 5$ for $n = 0, 1, 2, 3$. Note that $C(0) = 1$, not 0. In general, for $n \geq 1$, the following recurrence is evident:

$$C(n) = \sum_{i=1}^{n} C(i-1)C(n-1-i). \tag{23}$$

We can interprete $C(n)$ as the number of binary trees with exactly n nodes (Exercise). In terms of $P(n)$, we get a similar recurrence:

$$P(n) = \sum_{i=1}^{n-1} P(i)P(n-1-i) \tag{24}$$

where we define $P(0) = 0$. Thus $P(1) = P(2) = 1, P(3) = 2$.

This recurrence has an elegant solution using generating functions (see §VIII.9),

$$C(m) = \frac{1}{m+1}\binom{2m}{m}.$$

By Stirling's approximation,

$$\binom{2m}{m} = \Theta\left(\frac{4^m}{\sqrt{m}}\right).$$

So $C(m) = \Theta(4^m m^{-3/2})$ grows exponentially and there is no hope to find the optimal parenthesis tree by enumerating all parenthesis trees.

¶27. Matrix Chain Product. An instance of the parenthesis problem is the **matrix chain product** problem: given a sequence

$$A_1, \ldots, A_n$$

of rectangular matrices where A_i is $a_{i-1} \times a_i$ ($i = 1, \ldots, n$), we want to compute the chain product

$$A_1 A_2 \cdots A_n$$

in the cheapest way. The sequence $(a_0, a_1, \ldots, a_n)$ of numbers is called the **dimension** of this chain product expression.

We need to clarify the cost model. Using associativity of matrix products, each method of computing this product corresponds to a distinct parenthesis tree on $(A_1, \ldots, A_n)$. For instance,

$$((A_1 A_2) A_3), \quad (A_1 (A_2 A_3))$$

are the two ways of multiplying 3 matrices. We assume that it costs pqr to multiply a $p \times q$ matrix by a $q \times r$ matrix. Hence if the dimension of the chain product $A_1 A_2 A_3$ is (a_0, a_1, a_2, a_3), the first method to multiply these three matrices above costs

$$a_0 a_1 a_2 + a_0 a_2 a_3 = a_0 a_2 (a_1 + a_3)$$

while the second method costs

$$a_0 a_1 a_3 + a_1 a_2 a_3 = a_1 a_3 (a_0 + a_2).$$

Letting $(a_0, \ldots, a_3) = (1, d, 1, d)$, these two methods cost $2d$ and $2d^2$, respectively. Hence the second method may be arbitrarily more expensive than the first.

Corresponding to the dimension $(a_0, \ldots, a_n)$ of a chain product instance, we define an triangular weight function $W(i, j, k)$ for $0 \le i < j < k \le n$ to reflect our complexity model:

$$W(i, j, k) := a_i a_j a_k.$$

This is what we called the "product weight function" in §2. The optimal method of computing a matrix chain product is reduced to the optimal parenthesis tree problem. We have seen an $O(n^3)$ solution to this problem.

The original problem of matrix chain product can be solved in two stages: first find the optimal parenthesis tree, based on just the dimension of the chain. Then use the parenthesis tree to order the actual matrix multiplications. The only creative part of this solution is the determination of the optimal parenthesization.

Remark: For the product weight function, $W(a_i, a_j, a_k) = a_i a_j a_k$, the optimal triangulation problem can be solved in $O(n \log n)$ time, using a sophisticated algorithm due to Hu and Shing [5]. Ramanan [8] gave an exposition of this algorithm, and presented an $\Omega(n \log n)$ lower bound in an algebraic decision tree.

Exercises

Exercise 5.1: Show that $C(n)$ is the number of binary trees on n nodes. HINT: Use the recurrence (1) and structural induction on the definition of a binary tree. ◊

Exercise 5.2: Work out the bijective correspondence between triangulations and parenthesis trees stated above. ◊

Exercise 5.3: Verify by induction that $C(m)$ has the claimed solution. ♢

Exercise 5.4: Solve the recurrence (1) for $C(n)$ by using the following observation: consider generating function

$$G(x) = \sum_{i=0}^{\infty} C(i)x^i = 1 + x + 2x^2 + 5x^3 + \cdots .$$

HINT: What can you say about the coefficient of x^n in the squared generating function $G(x)^2$? Write this down as a recurrence equation involving $G(x)$ Solve this quadratic equation.

♢

Exercise 5.5: i) Consider an abstract n-gon whose weight function is a product function, $W(i,j,k) = w_i w_j w_k$ for some sequence $w_1, \ldots, w_n$ of non-negative numbers. Call w_i the "weight" of vertex i. Let $(\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $\{1, \ldots, n\}$ such that

$$w_{\pi_1} \le w_{\pi_2} \le \cdots \le w_{\pi_n}.$$

Show that there exists an optimal triangulation T of P such that vertex π_1 of least weight is **connected to** π_2 and also to π_3 in T. [We say vertex i is **connected to** j **in** T if either ij or ji is in T or is an edge of the n-gon.]
HINT: Use induction on n. Call a vertex i **isolated** if it is not connected to another vertex by a chord in T. Consider two cases, depending on whether π_1 is isolated in T or not.

ii) (Open) Can you exploit this result to obtain a $o(n^3)$ algorithm for the matrix chain product problem? ♢

End Exercises

§6. Optimal Binary Trees

Suppose we store n keys

$$K_1 < K_2 < \cdots < K_n$$

in a binary search tree. The probability that a key K to be searched is equal K_i is $p_i \ge 0$, and the probability that K falls between K_j and K_{j+1} is $q_j \ge 0$. Naturally,

$$\sum_{i=1}^{n} p_i + \sum_{j=0}^{n} q_j = 1.$$

In our formulation, we do not restrict the sum of the p's and q's to be 1, since we can simply interpret these numbers to be "relative weights". But we do require the q_j, p_i's to be non-negative.

We want to construct an full[6] binary search tree T whose nodes are labeled by

$$q_0, p_1, q_1, p_2, \ldots, q_{n-1}, p_n, q_n \tag{25}$$

in symmetric order. Note that the p_i's label the internal nodes and q_j's label the leaves.

[6]This amounts to an extended binary search tree, as described in Lecture 3.

[FIGURE]

In a natural way, T corresponds to a binary search tree in which the internal nodes are labeled by $K_1, \ldots, K_n$. But for our purposes, the actual keys K_i are irrelevant: only the probabilities p_i, q_j are of interest. Each subtree $T_{i,j}$ $(1 \le i \le j \le n)$ of T corresponds to a binary search tree on the keys $K_i, \ldots, K_j$. We define the following **weight function**:

$$\begin{aligned} W(i-1,j) &:= q_{i-1} + p_i + q_i + \cdots p_j + q_j \\ &= q_{i-1} + \sum_{k=i}^{j} (q_k + p_k) \end{aligned}$$

for all $0 \le i \le j \le n$. Thus $W(i,i) = q_i$. The **cost** of T is given by

$$C(T) = W(0,n) + C(T_L) + C(T_R)$$

where T_L and T_R are the left and right subtrees of T. If T has only one node, then $C(T) = 0$, corresponding to the case where the node is labeled by some q_j. We say T is **optimal** if its cost is minimum. So the problem of **optimal search trees** is that of computing an optimal T, given the data in (1). Why is this definition of "cost" reasonable? Let us charge a unit cost to each node we visit when we lookup a key K. If K has the frequency distribution given by the probabilities p_i, q_j, then the expected charge to the root of T is precisely $W(i-1,j)$ if the leaves of T are $K_i, \ldots, K_j$. So $C(T)$ is the expected cost of looking up K in the search tree T.

¶28. Application. In constructing compilers for programming languages, we need a search structure for looking up if a given identifier K is a key word. Suppose $K_1, \ldots, K_n$ are the key words of our programming language and we have statistics telling us that an identifier K in a typical program is equal to K_i with probability p_i and lies between K_j and K_{j+1} with probability q_j. One solution to this compiler problem is to construct an optimal search tree for the key words with these probabilities.

¶29. Example. Assume that $(p_1, p_2, p_3) = (6, 1, 3)$ and the q_i's are zero. There are 5 possible search trees here (see figure 6). The optimal search tree has root labeled p_1, giving a cost of $6 + 2(3) + 3(1) = 15$. Note that the structurally "balanced tree" with p_2 at the root has a bigger cost of 19. Intuitively, we understand why it is better to have p_1 at the root – it has a much larger frequency than the other nodes.

Figure 6: The 5 possible binary search trees on (p_1, p_2, p_3).

Let us observe that the **dynamic programming principle** holds, *i.e.*, every subtree of $T_{i,j}$ $(1 \le i \le n)$ is optimal for its associated relative weights

$$q_{i-1}, p_i, q_i, \ldots, q_{j-1}, p_j, q_j.$$

Hence an obvious dynamic programming algorithm can be devised to find optimal search trees in $O(n^3)$ time. Exploiting additional properties of the cost function, Knuth shows this can be done in $O(n^2)$ time. The key to the improvement is due to a general inequality satisfied by the cost function, first clarified by F. Yao, which we treat next.

Exercises

Exercise 6.1: Describe the precise connection between the optimal search tree problem and the optimal triangularization problem. ♢

Exercise 6.2: Suppose the input frequencies are $(p_1, \ldots, p_n)$ (the q_i's are all zero). If the p_i's are distinct, Joe Quick has a suggestion: why not choose the largest p_i to be the root? Is this true for $n = 3$? Find the smallest n for which this is false, and provide a counter example for this n.

♢

Exercise 6.3: (Project) Collect several programs in your programming language X.
(a) Make a sorted list of all the key words in language X. If there are n key words, construct a count of the number of occurrences of these key words in your set of programs. Let $p_1, p_2, \ldots, p_n$ be these frequencies.
(b) Construct an optimum search tree for these key words (assuming q_i's are 0) these key words (assuming q_i's are 0).
(c) Construct from your programs the frequencies that a non-key word falls between the keywords, and thereby obtain $q_0, q_1, \ldots, q_n$. Construct an optimum search tree for these p's and q's. ♢

Exercise 6.4: The following class of recurrences was investigated by Fredman [3]:

$$M(n) = g(n) + \min_{0 \le k \le n-1} \{\alpha M(k) + \beta M(n-k-1)\}$$

where $\alpha, \beta > 0$ and $g(n)$ are given. This is clearly related to optimal search trees. We focus on $g(n) = n$.
(a) Suppose $\min\{\alpha, \beta\} < 1$. Show that $M(n) \sim \frac{n}{1-\min\{\alpha,\beta\}}$.
(b) Suppose $\min\{\alpha, \beta\} > 1$, $\log\alpha/\log\beta$ is rational and $\alpha^{-1} + \beta^{-1} = 1$. Then $M(n) = \Theta(n^2)$.
♢

Exercise 6.5: If the p_i's are all zero in the Optimal Search Tree problem, then the optimization criteria amounts to minimizing the external path length. Recall that the external path length of a tree whose leaves are weighted is equal to $\sum_u d(u)w(u)$ where u ranges over the leaves, with $w(u), d(u)$ denoting the weight and depth of u. Suppose we consider a **modified path length** of a leaf u to be $w(u)\sum_{i=0}^{d(u)} 2^{-i}$ (instead of $d(u)w(u)$). Solve the Optimal Search Tree under this criteria. REMARK: This problem is motivated by the processing of cartographic maps of the counties in a state. We want to form a hierarchical level-of-detail map of the state by merging the counties. After the merge of a pair of maps, we always simplify the result by discarding some details. If the weight of a map is the number of edges or vertices in its representation, then after a simplification step, we are left with half as many edges. ♢

Exercise 6.6: Consider the following generalization of Optimal Binary Trees. We are given a subdivision of the plane into simply connected regions. Each region has a positive weight. We want to construct a binary tree T with these regions as leaves subject to one condition: each internal node u of T determines a subregion R_u of the plane, obtained as the union of all the regions below u. We require R_u to be simply-connected. The cost of T is as usual the external path length (i.e., sum of the weights of each leaf multiplied by its depth).
(a) Show that this problem is NP-complete.
(b) Give provably good heuristics for this problem. ♢

End Exercises

§7. Weight Matrices

We reformulate the optimal search tree problem in an abstract framework.

Definition 1. *Let $n \geq 2$ be an integer. A* **triangular function** *W (of order n) is any partial function with domain $[0..n] \times [0..n]$ such $W(i,j)$ is defined iff $i \leq j$. We call W a* **weight matrix** *if it is a triangular function whose range is the set of non-negative real numbers. A quadruple (i, i', j, j') is* **admissible** *if*

$$0 \leq i \leq i' \leq j \leq j' \leq n.$$

We say W is **monotone** *if*

$$W(i',j) \leq W(i,j')$$

for all admissible (i,i',j,j'). The **quadrangle inequality for** *W for (i,i',j,j') is*

$$W(i,j) + W(i',j') \leq W(i,j') + W(i',j).$$

We say W is **quadrangular** *if it satisfies the quadrangular inequality for all admissible (i,i',j,j').*

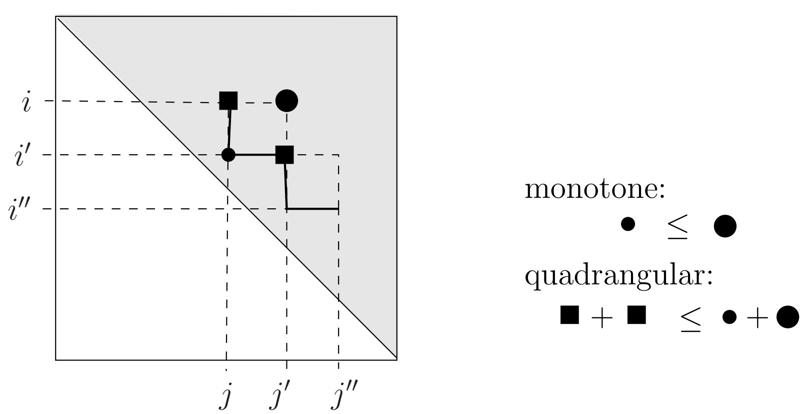


Figure 7: Monotone and quadrangular weight matrix.

It is sometimes convenient to write W_{ij} or $W_{i,j}$ instead of $W(i,j)$. If we view W_{ij} as the (i,j)-th entry of an n-square matrix W, then W is upper triangular matrix. Note that (i,i',j,j') is admissible iff the four points $(i,j),(i',j),(i,j'),(i',j')$ are all on or above the main diagonal of W (see Figure 7). Monotonicity and quadrangularity is also best seen visually (cf. Figure 7):

- Monotonic means that along any north-eastern path in the upper triangular matrix, the matrix values are non-decreasing.
- Quadrangularity means that for any 4 corner entries of a rectangle lying on or above the main diagonal, the south-west plus the north-east entries are not less than the sum of the other two.

Example: In the optimal search tree problem, the weight function W is implicitly specified by $O(n)$ parameters, *viz.*, $q_0, p_1, q_1, \ldots, p_n, q_n$, with

$$W(i,j) = \sum_{k=i-1}^{j} q_k + \sum_{k=i}^{j} p_k.$$

In this case, $W(i,j)$ can be computed in linear time from the q_k's and p_k's. The point is that, depending on the representation, $W(i,j)$ may not be available in constant time. The following is left as an exercise:

LEMMA 3. *The weight matrix for the optimal search tree problem is both monotone and quadrangular. In fact, the quadrangular inequality is an equality.*

DEFINITION 2. *Given a weight matrix W, its* **derived weight matrix** *is the triangular function*

$$W^* : [0..n]^2 \to \mathbb{R}_{\geq 0}$$

is defined as follows:

$$W^*(i,i) := W(i,i).$$

Assuming that $W^*(i,j)$ has been defined for all $j - i < \ell$, define

$$W^*(i, i+\ell) := W(i, i+\ell) + \min_{i<k\leq i+\ell} \{W^*(i,k-1) + W^*(k,i+\ell)\}.$$

Defining

$$W^*(i,j;k) := W(i,j) + W^*(i,k-1) + W^*(k,j), \tag{26}$$

*we call k an (i,j)-***splitter** *if $W^*(i,j) = W^*(i,j;k)$.*

Note: the literature (especially in operations research) describes the Monge property of matrices. This turns out to be the quadrangle inequality restricted to admissible quadruples (i,i',j,j') where $i' = i+1$ and $j' = j+1$.

EXERCISES

Exercise 7.1: (a) Computer the derived matrix of the following weight matrices:

$W_1 =$

1	1	1	1
	2	2	2
		3	3
			4

, $W_2 =$

1	2	1	2	1
	2	0	3	2
		1	0	1
			4	2
				2

.

(b) Suppose $W(i,j) = a_i$ for $i = j$ and $W(i,j) = 0$ for $i \neq j$. The a_i's are arbitrary constants. Succinctly describe the matrix W^*. ♢

Exercise 7.2: (Lemma 3) Verify that the weight matrix for the optimal search tree problem is indeed monotone and satisfies the quadrangular *equality*. ♢

Exercise 7.3: Write a program to compute the derivative of a matrix. It should run in $O(n^3)$ time on an n-square matrix. ♢

Exercise 7.4:
(a) Interpret the derived matrix for the optimal search tree problem.
(b) Does the derived matrix of a derived matrix have a realistic interpretation? ♢

Exercise 7.5: Generalize the concept of a triangular function W so that its domain is $[0..n]^k$ for any integer $k \geq 2$, and $W(i_1, \ldots, i_k)$ is defined iff $i_1 \leq i_2 \leq \cdots \leq i_k$. Then W is a **weight function** (of **order** n and **dimension** k) if it is triangular and has range over the non-negative real numbers. Formulate the "optimal k-gonalization" problem for an abstract n-gon. (This seeks to partition an n-gon into ℓ-gons where $3 \leq \ell \leq k$. Give a dynamic programming solution. ♢

End Exercises

§8. Quadrangular Inequality

The quadrangular inequality is central in the $O(n^2)$ solution of the optimal search tree problem. We will show two key lemmas.

Lemma 4. *If W is monotone and quadrangular, then the derived weight matrix W^* is also quadrangular.*

Proof. We must show the quadrangular inequality

$$W^*(i,j) + W^*(i',j') \leq W^*(i,j') + W^*(i',j), \qquad (0 \leq i \leq i' \leq j \leq j' \leq n). \tag{27}$$

First, we make the simple observation when $i = i'$ or $j = j'$, the inequality in equation (1) holds trivially.

The proof is by induction on $\ell = j' - i$. The basis, when $\ell = 1$, is immediate from the previous observation, since we have $i = i'$ or $j = j'$ in this case.

¶30. Case $i < i' = j < j'$: So we want to prove that $W^*(i,j) + W^*(j,j') \leq W^*(i,j') + W^*(j,j)$. Let $W^*(i,j') = W(i,j';k)$ and initially assume $i < k \leq j$. Then

$$\begin{array}{rcll} W^*_{i,j} + W^*_{j,j'} & \leq & [W_{i,j} + W^*_{i,k-1} + W^*_{k,j}] + W^*_{j,j'} & \text{(expanding } W^*_{i,j}\text{)} \\ & \leq & W_{i,j'} + W^*_{i,k-1} + [W^*_{k,j} + W^*_{j,j'}] & \text{(by monotonicity)} \\ & \leq & [W_{i,j'} + W^*_{i,k-1} + W^*_{k,j'}] + W^*_{j,j} & \text{(by induction)} \\ & = & W^*_{i,j'} + W^*_{j,j} & \text{(by choice of } k\text{).} \end{array}$$

In case $j < k \leq j'$, we would initially expand $W^*_{j,j'}$ above.

¶31. Case $i < i' < j < j'$: Let $W^*(i, j') = W(i, j'; k)$ and $W^*(i', j) = W(i', j; \ell)$ and initially assume $k \leq \ell$. Then

$$\begin{array}{rcll} W^*_{i,j} + W^*_{i',j'} & \leq & [W_{i,j} + W^*_{i,k-1} + W^*_{k,j}] + [W_{i',j'} + W^*_{i',\ell-1} + W^*_{\ell,j'}] & \text{(since } i < k \leq j, i' < \ell \leq j'\text{)} \\ & \leq & [W_{i,j'} + W_{i',j}] + W^*_{i,k-1} + W^*_{i',\ell-1} + [W^*_{k,j} + W^*_{\ell,j'}] & (W \text{ is quadrangular}) \\ & \leq & [W_{i,j'} + W_{i',j}] + W^*_{i,k-1} + W^*_{i',\ell-1} + [W^*_{k,j'} + W^*_{\ell,j}] & \text{(induction on } (k, \ell, j, j')\text{)} \\ & \leq & [W_{i,j'} + W^*_{i,k-1} + W^*_{k,j'}] + [W_{i',j} + W^*_{i',\ell-1} + W^*_{\ell,j}] & \\ & = & W^*_{i,j'} + W^*_{i',j} & \text{(by choice of } k, \ell\text{).} \end{array}$$

In case $\ell < k$, we can begin as above with the initial inequality $W^*(i, j) + W^*(i', j') \leq W^*(i, j; \ell) + W^*(i', j'; k)$. **Q.E.D.**

¶32. Splitting function K_W. The (i, j)-splitter k is not unique but we make it unique in the next definition by choosing the largest such k.

DEFINITION 3. *Let W be an weight matrix. Define the* **splitting function** *K_W to be a triangular function*

$$K_W : [0..n]^2 \to [0..n]$$

defined as follows: $K_W(i, i) = i$ and for $0 \leq i < j \leq n$,

$$K_W(i, j) := \max\{k : W^*(i, j) = W(i, j; k)\}.$$

We simply write $K(i, j)$ for $K_W(i, j)$ when W is understood. Once the function K_W is determined, it is a straightforward matter to compute the derived matrix of W The following is the key to a faster algorithm.

LEMMA 5. *If the derived weight matrix of W is quadrangular, then for all $0 \leq i \leq j < j$,*

$$K_W(i, j) \leq K_W(i, j + 1) \leq K_W(i + 1, j + 1).$$

Proof. By symmetry, it suffices to prove that

$$K(i, j) \leq K(i, j + 1). \tag{28}$$

This is implied by the following claim: if $i < k \leq k' \leq j$ then

$$W^*(i, j; k') \leq W^*(i, j; k) \quad \text{implies} \quad W^*(i, j + 1; k') \leq W^*(i, j + 1; k). \tag{29}$$

To see the implication, suppose equation (1) fails, say $K(i, j) = k' > k = K(i, j + 1)$. Then the claim implies $K(i, j + 1) \geq k'$, contradiction.

It remains to show the claim. Consider the quadrangular inequality for the admissible quadruple $(k, k', j, j + 1)$,

$$W^*(k, j) + W^*(k', j + 1) \leq W^*(k, j + 1) + W^*(k', j).$$

Adding $W(i, j) + W(i, j + 1) + W^*(i, k - 1) + W^*(i, k' - 1)$ to both sides, we obtain

$$W^*(i, j; k) + W^*(i, j + 1; k') \leq W^*(i, j + 1; k) + W^*(i, j; k').$$

This implies equation (1). **Q.E.D.**

¶33. Main result. The previous lemma gives rise to a faster dynamic programming solution for monotone quadrangular weight functions.

Theorem 6. *Let W be weight matrix such that $W(i,j)$ can be computed in constant time for all $1 \le i \le j \le n$, and its derived matrix W^* is quadrangular. Then its derived matrix W^* and the splitting function K_W can be computed in $O(n^2)$ time and space.*

Proof. We proceed in stages. In stage $\ell = 1, \ldots, n-1$, we will compute $K(i, i+\ell)$ and $W^*(i, i+\ell)$ (for all $i = 0, \ldots, n-\ell$). It suffices to show that each stage takes takes $O(n)$ time. We compute $W^*(i, i+\ell)$ using the minimization

$$W^*(i, i+\ell) = \min\{W(i, i+\ell; k) : K(i, i+\ell-1) \le k \le K(i+1, i+\ell)\}.$$

This equation is justified by the previous lemma, and it takes time $O(K(i+1, i+\ell) - K(i, i+\ell-1)+1)$. Summing over all $i = 1, \ldots, n-\ell$, we get the telescoping sum

$$\sum_{i=1}^{n-\ell} [K(i+1, i+\ell) - K(i, i+\ell-1) + 1] = n - \ell + K(n-\ell+1, n) - K(1, \ell) = O(n).$$

Hence stage ℓ takes $O(n)$ time. **Q.E.D.**

¶34. Remarks. We refer to [6] for a history of this problem and related work. The original formulation of the optimal search tree problem assumes p_i's are zero. For this case, T.C. Hu has an non-obvious algorithm that Hu and Tucker were able to show runs correctly in $O(n \log n)$ time. Mehlhorn [7] considers "approximate" optimal trees and show that these can be constructed in $O(n \log n)$ time. He describes a solution to the "approximate search tree" problem in which we dynamically change the frequencies; see "Dynamic binary search", (**SIAM J.Comp.**,8:2(1979)175–198). M. R. Garey gives an efficient algorithm when we want the optimal tree subject to a depth bound; see "Optimal Binary Search Trees with Restricted Maximum Depth, (**SIAM J.Comp.**,3:2(1974)101-110).

Exercises

Exercise 8.1: (a) Compute the optimal binary tree for the following sequence:

$$(q_0, p_1, q_1, \ldots, p_{10}, q_{10}) = (1, 2, 0, 1, 1, 3, 2, 0, 1, 2, 4, 1, 3, 3, 2, 1, 2, 5, 1, 0, 2).$$

(b) Compute the optimal binary tree for the case where the q's are the same as in (a), namely,

$$(q_0, q_1, \ldots, q_{10}) = (1, 0, 1, 2, 1, 4, 3, 2, 2, 1, 2)$$

and the p's are 0. ◇

Exercise 8.2: It is actually easy to give a "graphical" proof of lemma 5. In the figure 8, this amounts to showing that if $A + a \ge B + b$ then $A' + a' \ge B' + b'$.

◇

Exercise 8.3: If W is monotone and quadrangular, is W^* monotone? ◇

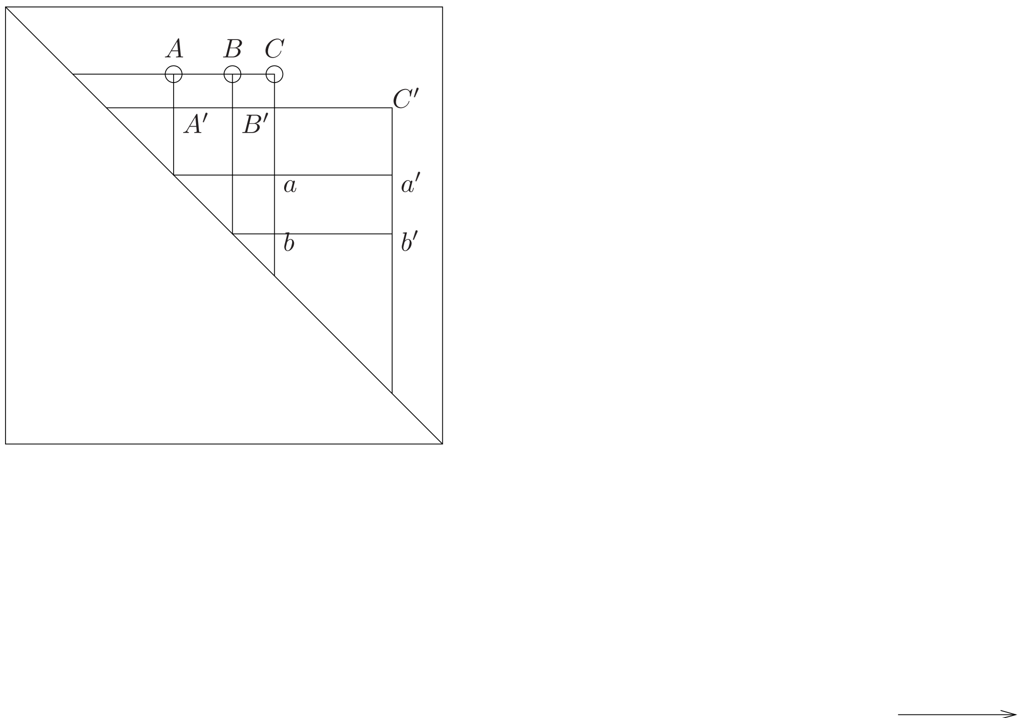


Figure 8: Derived weight matrix.

Exercise 8.4: Consider a binary search tree that has this shape (essentially a linear list):

Show that the following set of inequalities is necessary and sufficient for the above search tree to be optimal:

$$\begin{array}{rcll} p_2+q_2 & \ge & p_1+q_0 & (E_2) \\ p_3+q_3 & \ge & p_2+q_1+p_1+q_0 & (E_3) \\ \cdots & & & \\ p_n+q_n & \ge & p_{n-1}+q_{n-2}+p_{n-2}+\cdots+p_1+q_0 & (E_n) \end{array}$$

HINT: use induction to prove sufficiency.
Remark: So search trees with such shapes can be verified to be optimal in linear time. In general, can an search tree be verified to be optimal in $o(n^2)$ time? ♢

Exercise 8.5: (a) Generalize the above result so that all the internal nodes to the left of the root are left-child of its parent, and all the internal nodes to the right of the root are right-child of its parent. (b) Can you generalized this to the case where all the internal nodes lie on one path (ignoring directions along the tree edges – the path first traverses up the tree to the root and then down the tree again). ♢

Exercise 8.6: Given a sequence $a_1,\ldots,a_n$ of real numbers. Let $A_{ij}=\sum_{k=i}^{j} a_k$, $B_{ij}=\min\{A_{kj} : k=i,\ldots,j\}$ and $B_j = B_{1j}$. Compute the values $B_1,\ldots,B_n$ in $O(n)$ time. ♢

End Exercises

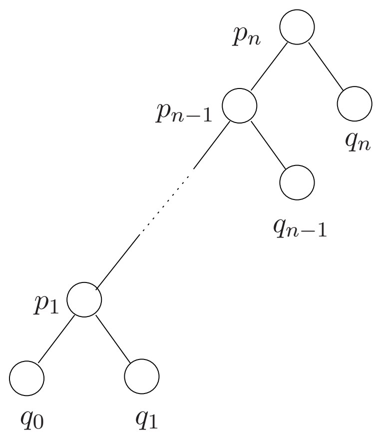


Figure 9: Linear list search tree.

§9. Conclusion

This chapter shows the versatility of the on dynamic programming approach to a variety of problems. A serious drawback of dynamic programming is its high polynomial cost: $O(n^k)$ for $k \geq 2$, in both time and space may not be practical in some applications. Hence there is interest in exploiting "sparsity conditions" when they occur. Sometimes, the implicit matrix to be searched has special properties (Monge conditions). See the survey of Giancarlo [4] for such examples.

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