Homework 1 Solutions
Fundamental Algorithms, Fall 2005, Professor Yap

Due: Thu Oct 6, in class.
HOMEWORK with SOLUTION, prepared by Instructor and T.A.s

INSTRUCTIONS:

- Please read questions carefully. When in doubt, please ask.
- There are links from the homework page to the old homeworks from previous classes, including solutions. Feel free to study these.

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1. (10+15 Points)
In the first lecture, we learned about the comparison tree model for studying sorting-like problems, and in particular about the information-theoretic lower bound.
(a) Give a lower bound on the number of comparisons that is necessary to sort $1,000,000$ elements.
(b) Give a good upper bound for sorting $1,000,000$ elements.

HINT: use an actual Mergesort as your basis for this problem. You may use our "exact analysis" (no big-Oh's) for the recurrences $T(n) = n + 2T(n/2)$, but be sure that your base case is correct.

Solution:
(a) Around 1.7×10^7. Given n elements to sort, the minimum number of leaf nodes in the comparison tree model is $n!$, so the height of the tree h is $h \geq \lg(n!)$. Using Stirling's formula and $2^{19} < 10^6 < 2^{20}$,

$$h \geq \lg(\sqrt{2\pi n}(n/e)^n) \geq n(\lg n - \lg e) \geq 10^6(19-2) = 1.7 \times 10^7$$

We could get more accurate approximation if we use a calculator. Good approximation is from 1.7×10^7 to 1.84×10^7.
(b) Around 2.0×10^7. We can use the Mergesort for the upper bound with the proper base case being $T(n) = 0$ for $n < 2$.

$$T(n) = n + 2T(n/2) = 2n + 2^2T(n/2^2) = \cdots = in + 2^iT(n/2^i)$$

Let $i = \lfloor \lg n \rfloor$, then $n/2^i < 2$ and $T(n) = n \lfloor \lg n \rfloor < 2^{20} \times 20$. Note that exact analysis of recurrrence is only correct when n is the power of 2.

2. (10 Points for each part)
Show the following by direct inequality arguments (no calculus):
(a) For all $k < k'$, $n^k = \mathcal{O}(n^{k'})$ and $n^k \neq \Omega(n^{k'})$.
(b) For all $k > 0$, show that $\lg n = \mathcal{O}(n^k)$.

HINT: since $\lg n = \Theta(H_n)$ where H_n is the Harmonic number, it is sufficient to show that $H_n = \mathcal{O}(n^k)$. See Fall 2001, hw2 solution for the proof that $H_n = \mathcal{O}(n^{1/2})$. Generalize this to any k of the form $k = 1/m$ where m is a natural number.
(c) Conclude from (a) and (b) that for all $k > 0$,

$$\lim_{n\to\infty} \frac{\lg n}{n^k} = 0.$$

Solution:
(a) Since $0 < n^k = n^{k-k'}n^{k'} < Cn^{k'}$ for C=1 and n > 1, $n^k = \mathcal{O}(n^{k'})$. Moreover, $n^k = o(n^{k'})$. (Note $\lim_{n\to\infty} \frac{n^k}{n^{k'}} = 0$)
If $n^k = \Omega(n^{k'})$, then for some $C > 0$, $Cn^k \geq n^{k'} \geq 0$ (ev.). Thus $C \geq n^{k'-k}$ (ev.). This is a

contradiction since $n^{k'-k}$ is unbounded. So $n^k \neq \Omega(n^{k'})$.
(b)

$$\begin{aligned}
H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\
&\leq \sum_{i=1}^{\lfloor n^{\frac{1}{m}} \rfloor} \frac{1}{i} + \sum_{i=\lfloor n^{\frac{1}{m}} \rfloor + 1}^{n^{\lfloor \frac{2}{m} \rfloor}} \frac{1}{i} + \sum_{i=\lfloor n^{\frac{2}{m}} \rfloor + 1}^{\lfloor n^{\frac{3}{m}} \rfloor} \frac{1}{i} + \cdots + \sum_{i=\lfloor n^{\frac{m-1}{m}} \rfloor + 1}^{n^{\frac{m}{m}}} \frac{1}{i} \\
&\leq \sum_{i=1}^{n^{\frac{1}{m}}} \frac{1}{i} + \sum_{i=n^{\frac{1}{m}}}^{n^{\frac{2}{m}}} \frac{1}{i} + \sum_{i=n^{\frac{2}{m}}}^{n^{\frac{3}{m}}} \frac{1}{i} + \cdots + \sum_{i=n^{\frac{m-1}{m}}}^{n^{\frac{m}{m}}} \frac{1}{i} \\
&\leq n^{\frac{1}{m}} + n^{\frac{1}{m}} + n^{\frac{1}{m}} + \cdots + n^{\frac{1}{m}} = mn^{\frac{1}{m}} = \mathcal{O}(n^k)
\end{aligned}$$

(c) For all $k > 0$, there exists some k' such that $k > k' > 0$. Since $\lg n = \mathcal{O}(n^{k'})$ and $n^{k'} = o(n^k)$, $\lg n = o(n^k)$. ¿From the definition of small oh, $\lim_{n\to\infty} \frac{\lg n}{n^k} = 0$.

3. (15 Points)
Use the Rote Method to solve the following recurrence:

$$T(n) = n^4 + 4T(n/2).$$

Be sure to indicate each of the EGVS steps. Choose your own initial conditions ("strong DIC").

Solution:

- Expand:

$$\begin{aligned}
T(n) &= n^4(1 + \frac{1}{4}) + 4^2 T(\frac{n}{2^2}) \\
&= n^4(1 + \frac{1}{4} + \frac{1}{4^2}) + 4^3 T(\frac{n}{2^3})
\end{aligned}$$

- Guess:

$$G(i) : T(n) = n^4 \sum_{k=0}^{i} (\frac{1}{4})^k + 4^{i+1} T(\frac{n}{2^{i+1}})$$

- Verify: Expand T(n) once again.

$$\begin{aligned}
T(n) &= n^4 \sum_{k=0}^{i} (\frac{1}{4})^k + 4^{i+1}[\frac{n^4}{2^{4i+4}} + 4T(\frac{n}{2^{i+2}})] \\
&= n^4 \sum_{k=0}^{i+1} (\frac{1}{4})^k + 4^{i+2} T(\frac{n}{2^{i+2}}) \\
&= G(i+1)
\end{aligned}$$

- Stopping Criteria: You can choose your own initial condition. Here, let us choose $T(n) = 0$ for $n < 2$. ¿From $\frac{n}{2^{i+1}} < 2$, we can set $i = \lfloor \lg n \rfloor - 1$. Then

$$T(n) = n^4 \sum_{k=0}^{\lfloor \lg n \rfloor - 1} (\frac{1}{4})^k = \frac{4n^4}{3}(1 - \frac{1}{4^{\lfloor \lg n \rfloor}})$$

4. (15 Points)
   Consider the recurrence
   $$T(n) = n^c + T(n/2) + T(n/3) + T(n/4).$$
   (a) Show by real induction that $T(n) = \mathcal{O}(n^c)$ when $c = 2$.
   (b) Suppose we slowly decrease c. At which value of c will the bound in part (a) become false?

   HINT: For part (a), do not worry about the initial conditions: just show that the induction can be carried through for n, assuming the result is true for all smaller n. Then by an appeal to some general theorem, we can say that your proof is actually rigorous.

   Solution:
   (a) Assume $T(n) \leq Kn^c$. Then

   $$\begin{aligned} T(n) &= n^c + T(n/2) + T(n/3) + T(n/4) \\ &\leq n^c + K(\tfrac{n}{2})^c + K(\tfrac{n}{3})^c + K(\tfrac{n}{4})^c && \text{(from induction hypothesis)} \\ &= n^c\{1 + K(\tfrac{1}{2^c} + \tfrac{1}{3^c} + \tfrac{1}{4^c})\} \\ &\leq Kn^c && (\text{if } 1 + K(\tfrac{1}{2^c} + \tfrac{1}{3^c} + \tfrac{1}{4^c}) \leq K\dagger) \end{aligned}$$

   ¿From †, $K(1 - \frac{1}{2^c} - \frac{1}{3^c} - \frac{1}{4^c}) \geq 1$, and it is satisfied when K≥1.8 for c=2.
   Now we have proven by induction that $T(n) \leq Kn^2$ for some K, hence $T(n) = \mathcal{O}(n^2)$.
   (b) ¿From the above, induction is valid when $\frac{1}{2^c} + \frac{1}{3^c} + \frac{1}{4^c} < 1$. Since LHS of the inequality increases as c decreases, c is bounded, say $c > c_0$. And c_0 is roughly 1.1. So the bound in part (a) is true for all $c > c0 \simeq 1.1$

5. (30 Points)
   Use our summation rules to give the Θ-order of the following sums:
   (a) $\sum_{i=1}^{n} H_i$
   (b) $\sum_{i=1}^{n} i^3 3^i$
   (c) $\sum_{i=1}^{n} i^3 2^{-i}$

   Solution:
   (a) $f(i) = H_i \leq H_{i/2} + 1 \leq 2H_{i/2}$ for $i > 4$. polynomial type. Hence $\Theta(nH(n))$ or $\Theta(n \log n)$.
   (b) Exponentially growing large. $\Theta(n^3 3^n)$.
   (c) Exponentially growing small. $\Theta(1)$.

6. (50 Points)
   Use the Master Theorem to solve the following recurrences. When the Master Theorem is not directly applicable, you should use it to give as tight an upper and lower bound you can. When using the Master Theorem, you must justify the case $(0, -1$ or $+1)$ that you are using.
   (a) $T(n) = 3T(n/25) + \log^3 n$
   (b) $T(n) = 8T(n/2) + n^3 \log^3 n$
   (c) $T(n) = T(\sqrt{n}) + n$
   (d) $T(n) = 25T(n/3) + (n/\log n)^3$
   (e) $T(n) = 9T(n/3) + n^2/\log\log n$
   HINT: for part (c), try to transform into a form for which the Master theorem is applicable.

   Solution:
   (a) $T(n) = \Theta(n^{\log_{25} 3})$.
   $f(n) = \log^3 n = \mathcal{O}(n^{\log_{25} 3 - \epsilon})$. This is the case -1, so $T(n) = \Theta(n^{\log_{25} 3})$.
   (b) Since $f(n) = \Omega(n^3)$ but $f(n) \neq \Omega(n^{3+\epsilon})$, we cannot apply Master Theorem.
   For upper bound, $T(n) \leq 8T(n/2) + n^{3+\epsilon} = \Theta(n^{3+\epsilon})$, hence $T(n) = \mathcal{O}(n^{3+\epsilon})$.
   For lower bound, $T(n) \geq 8T(n/2) + n^3 = \Theta(n^3 \log n)$. However, we can get better lower bound merely by looking at the recurrence equation. $T(n) = 8T(n/2) + n^3 \log^3 n \geq n^3 \log^3 n = \Theta(n^3 \log^3 n)$. Hence $T(n) = \Omega(n^3 \log^3 n)$.
   (c) $T(n) = \Theta(n)$.
   Let $n = 2^N$ and $t(N) = T(2^N)$. Then $t(N) = t(N/2) + 2^N$. It satisfies the regularity condition $af(N/b) = 2^{N/2} \leq c2^N$ for say c=0.5. So it is the case +1 and $T(n) = t(N) = \Theta(2^N) = \Theta(n)$.

(d) $T(n) = \Theta((n/\log n)^3)$.
$f(n) = \Omega(n^{\log_3 25+\epsilon})$. Regularity condition is satisfied when $c = \frac{26}{27}$ because $\lim_{n\to\infty} af(n/b)/f(n) = \lim_{n\to\infty} 25(\frac{1}{3}\frac{\log n}{\log n - \log 3})^3 = \frac{25}{27}$. So this is the case +1. Hence $T(n) = \Theta((n/\log n)^3)$.
(e) $f(n) = \mathcal{O}(n^2)$ but $f(n) \neq \mathcal{O}(n^{2-\epsilon})$. We cannot use Master Theorem.
For upper bound, $T(n) \leq 9T(n/3) + n^2 = \Theta(n^2 \log n)$, hence $T(n) = \mathcal{O}(n^2 \log n)$.
For lower bound, $T(n) \geq 9T(n/3) + n^{2-\epsilon} = \Theta(n^2)$, hence $T(n) = \Omega(n^2)$.