Homework 1

Fundamental Algorithms, Fall 2005, Professor Yap

Due: Thu Oct 6, in class.

INSTRUCTIONS:

- Please read questions carefully. When in doubt, please ask.
- There are links from the homework page to the old homeworks from previous classes, including solutions. Feel free to study these.

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1. (10+15 Points)
   In the first lecture, we learned about the comparison tree model for studying sorting-like problems, and in particular about the information-theoretic lower bound.
   (a) Give a lower bound on the number of comparisons that is necessary to sort $1,000,000$ elements.
   (b) Give a good upper bound for sorting $1,000,000$ elements.

   HINT: use an actual Mergesort as your basis for this problem. You may use our "exact analysis" (no big-Oh's) for the recurrences $T(n) = n + 2T(n/2)$, but be sure that your base case is correct.

2. (10 Points for each part)
   Show the following by direct inequality arguments (no calculus):
   (a) For all $k < k'$, $n^k = \mathcal{O}(n^{k'})$ and $n^k \neq \Omega(n^{k'})$.
   (b) For all $k > 0$, show that $\lg n = \mathcal{O}(n^k)$.

   HINT: since $\lg n = \Theta(H_n)$ where H_n is the Harmonic number, it is sufficient to show that $H_n = \mathcal{O}(n^k)$. See Fall 2001, hw2 solution for the proof that $H_n = \mathcal{O}(n^{1/2})$. Generalize this to any k of the form $k = 1/m$ where m is a natural number.
   (c) Conclude from (a) and (b) that for all $k > 0$,
   $$\lim_{n\to\infty} \frac{\lg n}{n^k} = 0.$$

3. (15 Points)
   Use the Rote Method to solve the following recurrence:
   $$T(n) = n^4 + 4T(n/2).$$
   Be sure to indicate each of the EGVS steps. Choose your own initial conditions ("strong DIC").

4. (15 Points)
   Consider the recurrence
   $$T(n) = n^c + T(n/2) + T(n/3) + T(n/4).$$
   (a) Show by real induction that $T(n) = \mathcal{O}(n^c)$ when $c = 2$.
   (b) Suppose we slowly decrease c. At which value of c will the bound in part (a) become false?

   HINT: For part (a), do not worry about the initial conditions: just show that the induction can be carried through for n, assuming the result is true for all smaller n. Then by an appeal to some general theorem, we can say that your proof is actually rigorous.

5. (30 Points)
   Use our summation rules to give the Θ-order of the following sums:
   (a) $\sum_{i=1}^{n} H_i$
   (b) $\sum_{i=1}^{n} i^3 3^i$
   (c) $\sum_{i=1}^{n} i^3 2^{-i}$

6. (50 Points)
Use the Master Theorem to solve the following recurrences. When the Master Theorem is not directly applicable, you should use it to give as tight an upper and lower bound you can. When using the Master Theorem, you must justify the case $(0, -1 \text{ or } +1)$ that you are using.
(a) $T(n) = 3T(n/25) + \log^3 n$
(b) $T(n) = 8T(n/2) + n^3 \log^3 n$
(c) $T(n) = T(\sqrt{n}) + n$
(d) $T(n) = 25T(n/3) + (n/\log n)^3$
(e) $T(n) = 9T(n/3) + n^2/\log\log n$
HINT: for part (c), try to transform into a form for which the Master theorem is applicable.