Homework 1

Fundamental Algorithms, Spring 2003, Professor Yap

Due: Mon Feb 10, in class

INSTRUCTIONS:

- Please read questions carefully. When in doubt, please ask.
- There are links from the homework page to the old homeworks from two previous classes, including solutions. Feel free to study these.

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1. (10 Points) Let $p(x) = 3x^3 - 1000x^2 + 1$. Prove the following statement: $p(x) = \Theta(x^3)$. NOTE: You should normally break up a Θ-statement into an upper bound statement and a lower bound statement.

2. (15 Points)
(a) What is the relation between these two statements: $f(n) \neq O(n^2)$ and $f(n) = \Omega(n^2)$.
(b) Construct an example of $f(n)$ for which these two statements are not equivalent.

3. (20 Points) This question should be done without resort to Calculus.
(a) Show that $H_n \to \infty$ as $n \to \infty$.
(b) Show that $H_n = O(n^{1/k})$ for all positive integer $k \geq 2$. HINT: Break the summation into k parts – this is similar to an argument in Section 6 of Lecture 2.
(c) Conclude from (b) that $\log n = O(n^\alpha)$ for all real $\alpha > 0$.

4. (10 Points) You have been asked to update Java's standard class "BigInteger" that perform arithmetic and other operations on arbitrarily large integers. You first determined that the old implementation of the multiplication algorithm takes time $T_0(n) = 2n^2 + 20n + 10$ for all $n \geq 1$ (this is, of course, an implementation of the High School Algorithm). Since you learned about Karatsuba, you decided to implement it, and found that its running time is $T_1(n) = 10n^\alpha + 20n + 60$ for all $n \geq 1$, where $\alpha = \lg 3 < 1.58$. How can you take advantage of these two multiplication algorithms for your next release of Java's "BigInteger" class? NOTE: You should do some numerical calculations with $T_0(n)$ and $T_1(n)$ using perhaps a pocket calculator.

5. (20 Points) Use the Rote Method to solve the following recurrence: $T(n) = 8T(n/2) + n$. The method involves 4 steps (Expand, Guess, Verify, Stop). Make sure that each step is clearly marked and explained. Be sure to tell us what initial condition you choose.

6. Let $T(n) = 10T(n/3) + n$.
(a) Use Real Induction to show that $T(n) = O(n^\alpha)$ when $\alpha = 3$.
(b) By examining your proof in (a), find the best possible value for α such that your proof will still work.