Homework 2

Fundamental Algorithms, Fall 2001, Professor Yap

DUE: Mon Oct 8, in class

NOTICE:

- Some students have reasons for extensions – either for compassionate reasons (especially related to the recent events), for religious holidays, or others. Please come to see me for this.
- We suggest trying to going over the homework before recitation, so that you can ask relevant questions.

1. Understanding the function $\log(x)$ function is very important. Recall that $\ln(x)$ means log to the natural base e, and $\lg(x)$ is log to base 2.

   (a) Clearly, $\ln(x)$ goes infinity as x goes to infinity. But how can we prove this without calculus? In discrete math, the log function arises as the harmonic numbers $H_n = 1 + (1/2) + \cdots + (1/n)$. You should know that $H_n = \ln(n) + Theta(1)$ (see text). Thus, it is enough to show that H_n goes to infinity as n goes to infinity. Prove this.

   (b) But while $\ln(x)$ is unbounded, it is also a very slow growing function. For instance, $H_n = O(n^c)$ for any constant $c > 0$. Prove this in the case of $c = 1/2$.

   HINTS: Please read Appendix A.2 for ideas for how to estimate sums. For part (a), we suggest that you assume $n = 2^k$ for some $k \geq 2$ and group the summation of H_n into k groups. For part (b), split the sum of H_n into two parts, the first part with about $\sqrt(n)$ terms.

2. Suppose we have two coins, one fair and one fake. The fair coin, has equal probability of showing head or tail. The fake coin is very biased: it always show head. Let C be a random coin, obtained by picking the fair or fake coin with equal probability. So, C is a random object with $Pr\{C = fair\ coin\} = 1/2$. Note: you may refer to the Textbook, page 1105 for a simpler version of this problem. Let B be the event $\{C = \textit{biased coin}\}$, and for any natural number n, let A_n be the event $\{n \text{ flips of } C \text{ yields head every time }\}$.

   (a) Compute the probability $\Pr(B|A_n)$.

   (b) What is the minimum number n of flips needed to determine whether C is fair or fake with 99% certainty? We want a numerical value for n, but show the working (probably using a hand calculator).

3. What is the probability space in the analysis of Quicksort? That is, you must describe a finite set S and tell us for each $w \in S$, tell us how to determine its probability $\Pr(w)$. Each w corresponds to a "experiment", or a complete run of the Quicksort algorithm. Illustrate the general describe with a special case $n = 4$. What is $|S|$ when $n = 4$? Give upper and lower bounds on the size of $|S|$.

4. Give another proof that Randomize-in-Place(A) is correct, using the formula in C.2-6 (p.1106)

   HINT: let $\sigma = (x_1, ..., x_n)$ be any permutation of $[1..n]$. Show that A_i be the event $\{A[i] = x_i\}$. Prove that the probablity that the array A represents σ is $1/n!$. Again, use the formula in C.2-6 (p.1106).

5. Exercise 5.3-5, page 105 (see also p.101). This exercise is to prove that the probability that all elements in the array P are unique at the end of the Permute-by-Sorting algorithm is at least $1-(1/n)$. HINT: use the formula in C.2-6 (p.1106), which we briefly mentioned class. The binomial series and the series for $e^x = 1 + x + x^2/2! + \cdots$ may be useful.

ADDITIONAL QUESTIONS (not graded, but we will sketch answers)

1. Generalize the proof above to show that $H_n = O(n^c)$ for any $c > 0$.
2. Exercise A.2-1 (p. 1067)
3. Exercise A-1 (p. 1069)