1. Question 3.1-2 page 50 (15 points)

Show that for any real constants a and b, where b > 0
$(n+a)^b = \Theta(n^b)$

Solution:
To show $(n+a)^b = \Theta(n^b)$, we need to find constants c_1, c_2, n_0 such that

$$0 <= c_1 n^b <= (n+a)^b <= c_2 n^b$$

for all $n >= n_0$, where c1 and c2 are positive.
Note that

$$n+a <= n+|a| <= 2n, \text{ when } n >= |a|$$

and

$$n+a >= n-|a| >= n/2, \text{ when } n >= 2|a|$$

So we have

$$0 <= n/2 <= n+a <= 2n, \text{ when } n >= 2|a|$$

Since b > 0, when 1 <= n/2 <= n+a <= 2n

$$0 <= (n/2)^b <= (n+a)^b <= (2n)^b, \text{ when } n >= 2|a|$$

i.e.

$$0 <= (1/2)^b n^b <= (n+a)^b <= 2^b n^b, \text{ when } n >= 2|a|$$

Let

$c_1 = (1/2)^b$
$c_2 = 2^b$
$n_0 = 2|a|$

we have $0 <= c_1 n^b <= (n+a)^b <= c_2 n^b$, so $(n+a)^b = \Theta(n^b)$. **Q.E.D.**

Note:
(1) a could be negative
(2) c_1, c_2 are positive, 0 is not allowed.
(3) c_1, c_2, n_0 are constants, so variable n is not allowed to appear in the expression of c_1, c_2, n_0.
(4) To prove this problem, you have to EXPLICITLY show the constants c_1, c_2, n_0.

2. Question 3-2 page 58 (20 points)

Solution:

	A	B	O	Ω	Θ
a	$\lg_k n$	n^ε	Yes	No	No
b	n^k	c^n	Yes	No	No
c	$n^{1/2}$	$n^{\sin n}$	No	No	No
d	2^n	$2^{n/2}$	No	Yes	No
e	$n^{\lg c}$	$c^{\lg n}$	Yes	Yes	Yes
f	$\lg(n!)$	$\lg(n^n)$	Yes	Yes	Yes

a) easy
b) easy
c) We know $-1 <= \sin n <= 1$. For some n_0, any $n > n_0$, we can't determine the relationship between 1 / 2 and n.
d) easy

e) $\lg(n^{\lg c}) = \lg(c^{\lg n}) = \lg n \lg c$
f) Sterling formula: $n! = (2\pi n)^{1/2}(n/e)^n(1+\theta(1/n))$ **Q.E.D.**

3. Question 4.3-2 page 75 (10 points)

Solution:

For $T(n) = 7T(n/2) + n^2$, we have

$a = 7, b = 2, f(n) = n^2$ and

$w(n) = n^{\log_b a} = n^{\log_2 7}$

For $T'(n) = aT'(n/4) + n^2$, we have

$w'(n) = n^{\log_4 a}$

Since A' is asymptotically faster than A, we have

$n^{\log_4 a} < n^{\log_2 7} \Rightarrow \log_4 a < \log_2 7 \Rightarrow \log_4 a < \log_4 49 \Rightarrow a < 49$

So the largest integer value for a is 48. **Q.E.D.**

4. Question 4-1 (40 points)

Solution:

(a) $T(n) = 2T(n/2) + n^3$

Use the master theorem

$a = 2, b = 2, w(n) = n$ and $f(n) = n^3$

Clearly $f(n) = \Omega(w(n).n^\varepsilon)$, where $\varepsilon=2$. And it's easy to verify that $af(n/b) <= cf(n)$ for some constant $c < 1$. To see this, note that we need to choose c such that $2(n/2)^3<=cn^3$; obviously we can choose $c = ¼$. Hence

$T(n) = \Theta(n^3)$

(b) $T(n) = T(9n/10) + n$

Use the master theorem,

$a = 1, b =10/9, w(n) = n^0 = 1$ and $f(n) = n$

Clearly $f(n) = \Omega(w(n).n^\varepsilon)$, where $\varepsilon=1$. And it's easy to verify that $af(n/b) <= cf(n)$ for some constant $c < 1$ and all sufficiently large n, so

$T(n) = \Theta(n)$

(c) $T(n) = 16T(n/4) + n^2$

Use the master theorem,

$a = 16, b = 4, w(n) = n^2$ and $f(n) = n^2$

Clearly $f(n) = \Theta(w(n))$. So

$T(n) = \Theta(n^2\lg n)$

(d) $T(n) = 7T(n/3) + n^2$

Use the master theorem,

$a = 7, b = 3, w(n) = n^{\log_3 7}$ and $f(n) = n^2$

Clearly $f(n) = \Omega(w(n).n^\varepsilon)$, where $\varepsilon>0$. And it's easy to verify that $af(n/b) <= cf(n)$ for some constant $c < 1$ and all sufficiently large n, so

$T(n) = \Theta(n^2)$

(e) $T(n) = 7T(n/2) + n^2$

Use the master theorem,

$a = 7$, $b = 2$, $w(n) = n^{\log_2 7}$ and $f(n) = n^2$

Clearly $f(n) = O(w(n).n^{-\varepsilon})$, where $\varepsilon>0$. So

$T(n) = \Theta(n^{\log_2 7})$

(f) $T(n) = 2T(n/4) + n^{1/2}$

Use the master theorem,

$a = 2$, $b = 4$, $w(n) = n^{1/2}$ and $f(n) = n^{1/2}$

Clearly $f(n) = \Theta (w(n))$. So

$T(n) = \Theta(n^{1/2}\lg n)$

(g) $T(n) = T(n-1) + n$

Master theorem is not applicable, however, we can solve the equation directly as follows:

$T(n) = n + T(n-1) = n + (n-1) + T(n-2) = \ldots= n + (n-1) + (n-2) + \ldots + 2 + 1 = n(n+1)/2$,

So **$T(n) = \Theta(n^2)$**

(h) $T(n) = T(n^{1/2}) + 1$

Master theorem is not applicable, however, we can use "domain transformation" as follows:

Let $k = \log_2 n$, so that $n = 2^k$

Let $S(k) = T(2^k) = T(2^{k/2}) + 1 = S(k/2) + 1$

Use the master theorem to $S(k)$, we have

$w(k) = 1$

So $S(k) = \Theta(\lg k)$, and $T(n) = S(\lg n) =$ **$\Theta(\lg\lg n)$ Q.E.D.**

Additional Questions

1. Question 3.1-4, page 50

Solution:

(a) Since $0 <= 2^{n+1} <= C_1*2^n$, for any $n > n_0 > 0$, $C_1 > 2$, $O(2^n) = 2^{n+1}$

(b) Suppose $2^{2n} = O(2^n)$, then we need to find constant positve c1 and n0, such that $0 <= 2^{2n} <= C_1*2^n$. Obvious, there is no such constant C_1, such that $C_1 > 2^n$, for any $n > n_0$, so $2^{2n} <> O(2^n)$. **Q.E.D.**

2. Question 3.2-3, page 57

Prove that $\lg(n!) = \Theta(n\lg n)$

Solution:

According to sterling's approximation,

$n! = (2\pi n)^{1/2}(n/e)^n(1+\Theta(1/n))$

So $\lg n! = n\lg n +(1/2)*\lg n – n\lg e + \lg(2\pi)^{1/2} + \lg(1+\Theta(1/n))$

Choose $c_1=1/2$, $c_2 = 3$, $n_0 = e^2$, we have

$0 <= c_1 n\lg n <= \lg(n!) <= c_2 n\lg n$

is hold for any n >= n0

So $\lg(n!) = \Theta(n\lg n)$ **Q.E.D.**

3. Question 4.3-1, page 75

Solution:

(a) $T(n) = 4T(n/2) + n$

Since $w(n) = n^2$ and $f(n) = n$, $f(n) = O(w(n)n^{-\varepsilon})$, where $\varepsilon = 1$

So $T(n) = \Theta(n^2)$ **Q.E.D.**

(b) $T(n) = 4T(n/2) + n^2$

Since $w(n) = n^2$, $f(n) = n^2$, we have $f(n) = \Theta(w(n))$

So $T(n) = \Theta(w(n)\lg n) = \Theta(n^2 \lg n)$ **Q.E.D.**

(c) $T(n) = 4T(n/2) + n^3$

Since $w(n) = n^2$, $f(n) = n^3$, we have $f(n) = \Omega(w(n)n^{\varepsilon})$, where $\varepsilon = 1$. And it's easy to verify that $af(n/b) <= cf(n)$ for some constant $c < 1$ and all sufficiently large n, so

$T(n) = \Theta(n^3)$ **Q.E.D.**