Sample Solution for Homework II

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1. **Question:** The following recurrence actually arise in the analysis of a famous fast algorithm for multiplying binary numbers:

$$T(x) \;=\; 3T(x/2) \;+\; x$$

if $x > 1$, otherwise $T(x) = 0$. Prove that $T(x) = \Theta(x^{\lg 3})$.

Solution: Let us expand the recurrence repeatedly, as follows:

$$\begin{aligned}
T(x) &= 3\,T(x/2) \;+\; x \\
&= 3\,(3\,T(\frac{x}{2^2}) \;+\; \frac{x}{2}) \;+\; x \\
&= 3^2\,T(\frac{x}{2^2}) \;+\; x[1 \;+\; (3/2)] \\
&= 3^3\,T(\frac{x}{2^3}) \;+\; x[1 \;+\; (3/2) \;+\; (3/2)^2] \\
&\cdots \\
&= 3^i\,T(\frac{x}{2^i}) \;+\; x[1 \;+\; (3/2) \;+\; (3/2)^2 \;+ \cdots + (3/2)^{i-1}]
\end{aligned}$$

The expansion is possible so long as $\frac{x}{2^i} > 1$. Thus to find the number of times we can apply the recurrence is determined by the inequalities:

$$\frac{x}{2^{i-1}} \;>\; 1 \qquad (1)$$

$$\frac{x}{2^{i}} \;\le\; 1 \qquad (2)$$

Thus, combining these two inequalities, we have $2^{i-1} < x \le 2^i$. i.e. $i = \lceil \lg x \rceil$. For this value of i, the term involving T vanishes, by the initial condition, and we are left with a geometric series. Summing this series, we get

$$\begin{aligned}
T(x) &= x\frac{(3/2)^i \;-\; 1}{(3/2) \;-\; 1} \\
&= 2x[(3/2)^i \;-\; 1] \\
&= 2x(3/2)^{\lceil \lg x \rceil} \;-\; 2x
\end{aligned}$$

Noticing that, $\lg x - 1 \leq \lceil \lg x \rceil \leq \lg x + 1$ we immediately conclude that

$$(4/3)x^{\lg 3} - 2x \leq T(x) \leq 3x^{\lg 3} - 2x$$

This proves that $T(x) = \Theta(x^{\lg 3})$.

2. **Question:** Use real induction to give an exact solution for the following recurrence on real numbers:

$$T(x) = 2T(x/2) + x$$

if $x > 1$; otherwise $T(x) = 1$.

Solution: As in the approach in the previous problem and as shown in class, we get

$$T(x) = 2^i T(\frac{x}{2^i}) + i\, x$$

where $i = \lceil \lg x \rceil$. The value of $T(\frac{x}{2^i})$ is 1, by the initial condition. Substituting this, and the value of i, we get

$$T(x) = x \lceil \lg x \rceil + 2^{\lceil \lg x \rceil}$$

where the second term is between $(1/2)x$ and $2x$.

To prove this by real induction, we note that the above formula reduces to $T(x) = x + 2$, for $1 < x \leq 2$. Also, for these values of x, the recurrence can be applied only once, before the boundary condition applies, and this gives the same formula $x + 2$. This proves the base case for induction.

To complete the induction, we assume that the induction hypothesis holds for $x/2$, and prove it to hold for x. This approach works because, we can only half x a finite number of times before it reduces to the base case. Thus, we have

$$\begin{aligned}
T(x) &= 2T(x/2) + x \\
&= 2[(x/2)\lceil \lg(x/2) \rceil + 2^{\lceil \lg \frac{x}{2} \rceil}] + x \\
&= 2[(x/2)(\lceil \lg x \rceil - 1) + 2^{(\lceil \lg x \rceil - 1)}] + x \\
&= x \lceil \lg x \rceil + 2^{\lceil \lg x \rceil}
\end{aligned}$$

where we used the result that $\lceil (a-1) \rceil = \lceil a \rceil - 1$.

3. **Question:** Problem 3.8 (Chapter 3, page 72) of the text book. It says "briefly explain", which we will interprete to mean that you show the implicit constants in the O, Ω or Θ.

Solution:

- $f(n) = \log n^2$ and $g(n) = \log n + 5$.
  $f(n) = \Theta(g(n))$.

- $f(n) = \sqrt{n}$ and $g(n) = \log n^2$.
  $f(n) = \Omega(g(n))$.
- $f(n) = \log^2 n$ and $g(n) = \log n$.
  $f(n) = \Omega(g(n))$.
- $f(n) = n$ and $g(n) = \log^2 n$.
  $f(n) = \Omega(g(n))$.
- $f(n) = n \log n + n$ and $g(n) = \log n$.
  $f(n) = \Omega(g(n))$.
- $f(n) = 10$ and $g(n) = \log 10$.
  $f(n) = \Theta(g(n))$.
- $f(n) = 2^n$ and $g(n) = 10n^2$.
  $f(n) = \Omega(g(n))$.
- $f(n) = 2^n$ and $g(n) = 3^n$.
  $f(n) = O(g(n))$.