Basic Algorithms (V22.0310); Fall 2005; Yap

QUIZ ONE (with SOLUTION)

Oct 12, 2005

Instructions

- Please questions carefully.

1 Question 1

Find the pairs of statements that are contra-positives of each other (recall that contra-positive statements are equivalent).
(a) $A \Rightarrow B$
(b) $A \Rightarrow \neg B$
(c) $\neg A \Rightarrow B$
(d) $\neg A \Rightarrow \neg B$

SOLUTION: There are no such pairs. The contra-positive of $A \Rightarrow B$ is $\neg B \implies \neg A$. Both are equivalent to $(\neg A) \vee B$.

2 Question 2

Consider the following statement about two real functions f, g:

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})\,[(z > y) \wedge (f(z) > x \cdot g(z))] \tag{1}$$

(a) Give an example of f, g which makes the statement true.
(b) Give an example of f, g which makes the statement false.

SOLUTION:
The statement says that $f \neq O(g)$.
(a) We can choose $f(x) = x^2$ and $g(x) = x$.
(b) We can choose $f(x) = x$ and $g(x) = x^2$.

3 Question 3

Order these:

$$2^n, \quad n, \quad \lg^2 n, \quad \sqrt{n}\lg n, \quad n^{2/3}, \quad n!, \quad \pi^n.$$

SOLUTION:

$$\lg^2 n, \quad \sqrt{n}\lg n, \quad n^{2/3}, \quad n, \quad 2^n, \quad \pi^n, \quad n!.$$

Most of these are standard. The last one can be seen as follows: $n! > (3!)(4^{n-3}) \geq \pi^{n-3}$ for $n > 3$. Thus, $Cn! > \pi^n$ for $C = \pi^3$ and $n_0 = 3$.

4 Question 4

Prove that $4n^2 + 5n = O(n^2)$.

SOLUTION:

By definition, we want to find a $C > 0$ and $n_0 > 0$ such that for all $n \geq n_0$, we have $4n^2 + 5n \leq Cn^2$.

Note that $5n \leq n^2$ for $n \geq 5$. Hence $4n^2 + 5n \leq 4n^2 + n^2 = 5n^2$ for $n \geq 5$.

So we can choose $C = 5$ and $n_0 = 5$. This completes our proof.

5 Question 5

Compute the DFS of the following digraph G_0, starting from node 0:

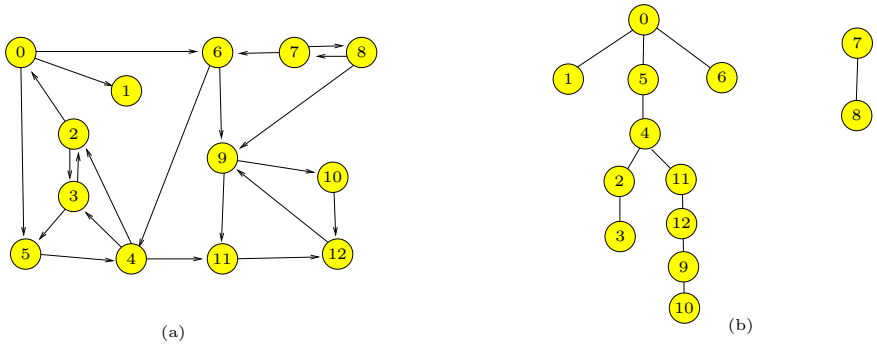


Figure 1: (a) Graph G_0, and (b) its DFS Tree from node 0

The DFS Tree solution is shown in (b).

NOTE: by convention, when searching the adjacent vertices, we assume we try the vertices in increasing numerical order.

6 Question 6

What are the strong components of G_0 above?

SOLUTION:

There are 4 strong components:

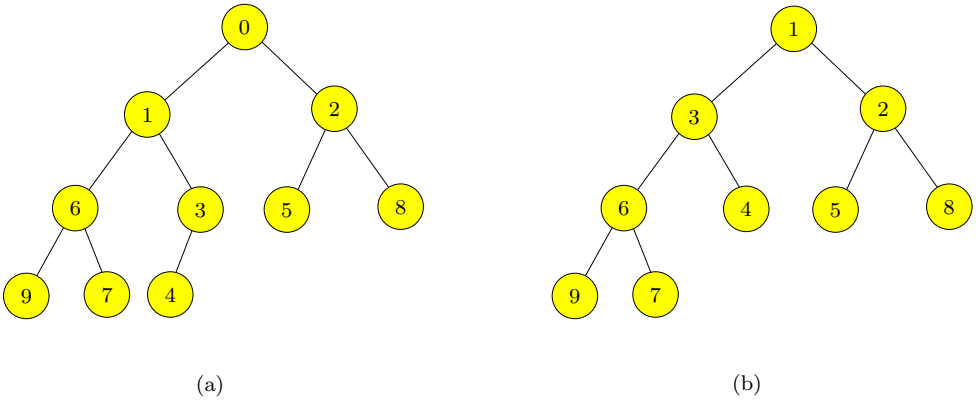
$$\{1\}, \{0, 2, 3, 4, 5, 6\}, \{9, 10, 11, 12\}, \{7, 8\}$$

7 Question 7

Show the heap after each of the the following insertions:

$$1, 9, 2, 4, 3, 5, 8, 6, 7, 0$$

Show the heap after we do a deleteMin on the final heap.
SOLUTION:
The heaps are shown in Figure 2.



(a) (b)

Figure 2: (a) At the end of all insertions, (b) After deleteMin

8 Question 8

Compute the MST of the following weighted graph:
SOLUTION:
The thick edges in (b) shows the MST. The total weight is 16.

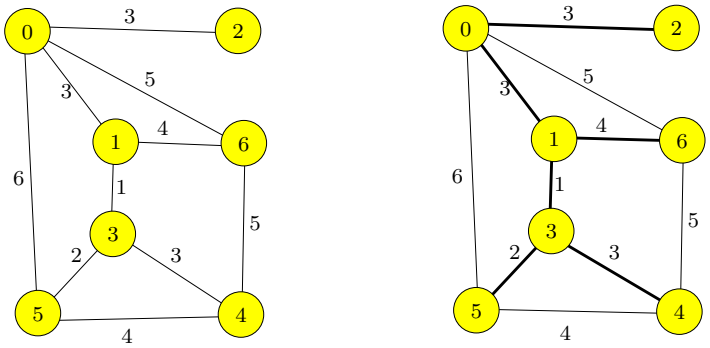


Figure 3: Weighted bigraph