Homework 5 Solutions

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The following solution were prepared by me (Chris), so if you find a typos email me at wu@cs.nyu.edu and not the professor. One comment is that these solutions are *complete* and contain many steps that are written for explanatory purposes so don't worry if you didn't write everything here.

Question 1

Part a)

Once you've set up the picture properly this question is pretty straight-forward. First, the box B' has width 3δ and height 2δ so its area is $6\delta^2$. Now, each point in S' that isn't part of the shortest distance pair has to be isolated in a ball of radius $\delta/2$. Each one of these circles has area $\pi(\delta/2)^2 = \frac{\delta^2 \cdot \pi}{4}$.

So you can fit at most $(6\delta^2)/(\frac{\delta^2 \cdot \pi}{4}) = \frac{24}{\pi}$ circles of this size in the box B'.

Part b)

Now if we actually try to fit these circles in the box, the most efficient way is to pack them 3 per row since the box has width 3δ and each circles has a diameter of δ. This means there will be 8 rows of circles. So the distance from the bottom to the top is 7 rows.

Question 2

Part a)

First, let's try and bound this by expansion:

$$\begin{aligned} T(n) &= 8T(n/2) + n^2 \\ &= 8(8T(n/4) + n^2/4) + n^2 \\ &= 8(8(8T(n/8) + n^2/16) + n^2/4) + n^2 \end{aligned}$$

So there are basically two types of terms coming from this recurrence. The first are the 8's in the first half. It's going to be 8^k for some k. How many? Well, there's one for each factor of 2 we can remove. So it's $8^{\log n}$ (KNOW THIS!). We can reduce this:

$$8^{\log n} = 2^{3^{\log n}} = 2^{\log n^3} = n^3$$

Now the second terms come for the n^2 part of the recurrence. If we write those out we get:

$$n^2 + 8 \cdot n^2/4 + 8^2 \cdot n^2/16 + \ldots$$

which is the same as

$$n^2 + 2 \cdot n^2 + 4 \cdot n^2 + \ldots = n^2(1 + 2 + 4 + \ldots)$$

This is something that you should familiar with. So we get $n^2 \cdot (2^{\log n+1} - 1)$. We'll hand wave this to another n^3. So we can conclude the whole thing is n^3.

Part b)

So we know that this will be $O(n^3)$ but we can choose any form that fits. I'll choose to prove that $T(n) \leq an^3 - (1/8)n^2$ where a is any positive number. By induction, let's assume this holds for all $k < n$.

$$\begin{aligned} T(n) &= 8T(n/2) + n^2 \\ &\leq 8(a(n/2)^3 - (1/8)n^2) + n^2 \\ &= an^3 \end{aligned}$$

The first line is just the definition. The second follows from induction. The third is just simplification. Notice that my choice of $(1/8)n^2$ wasn't random: it was chosen to nuke the n^2 term in the recurrence.

If we think about this, this might be a bit counterintuitive. I used $O(n^3)$ term with a negative term added to it. Isn't it harder to prove something with a stronger constraint? Well, the point is that by using a harder inductive hypothesis, I have a tighter inductive hypothesis to work with. So it balances out.

Part c)

First, let's write down the two recurrences

$$\begin{aligned} T(n) &= 3T(n/2) + n \\ T(n) &= 2T(n/3) + n \end{aligned}$$

I'll refer to them as recurrence one and recurrence two. So the numbers for the master method in the case of recurrence one are: $a = 3$, $b = 2$, $k = 1$ and $w = log_2 3 = 1.58$. So the master methods tell us it's $\Theta(n^{1.58})$.

Similarly, for the second recurrence we have: $a = 2$, $b = 3$, $k = 1$ and $w = log_3 2 = 0.63$. So we have $\Theta(n)$.

Part d)

Here the only difference is that $k = 2$ for both of them. This gives $\Theta(n^2)$ for both of them.