Basic Algorithms (V22.0310); Fall 2005; Yap

HOMEWORK 1

Date Due: Sep 21

SOLUTIONS to HW1

1 Question 1

(5 points) Please do Exercise 1, page 23, chapter 1. True or false: "In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first by w, and w is ranked first by m."

SOLUTION:

False. Suppose we have two men m, m' and two women w, w'. Let m rank w first; w rank m' first; m' rank w' first; and w' rank m first.

We see that such a pair as described by the claim does not exist.

2 Question 2

(5 points) Please do Exercise 2, page 23, chapter 1. True or false: "Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m ir ranked first by w, and w is ranked first by m. Then (m, w) must appear in every stable matching."

SOLUTION:

True. Suppose S is a stable matching that contains the pairs (m, w') and (m', w), for some m', w'. But clearly, (m, w) forms an instability pair, contradicting the stability of S.

3 Question 3

(5 points) Please do Exercise 3, page 23, chapter 1. We have two television networks, A and B. Network A wants to schedule a set $\{a_1, \ldots, a_n\}$ of shows into n time slots. Network B also wants to schedule a set $\{b_1, \ldots, b_n\}$ of shows into the same n slots. Let $\rho : \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \to \mathbb{R}$ denote the ratings of each show (no 2 shows have the same rating). If a_i and b_j are scheduled to the same slot, then we say network A wins this slot if $\rho(a_i) > \rho(b_j)$; otherwise network B wins this slot.

Does there exist a pair of schedules for networks A and B that is stable in the following sense: no network can unilaterally change its scheduling to win more slots.

SOLUTION: Solution:

We claim that there exists a rating such that no stable scheduling exists.

The scheduling of network A may be represented by a bijection:

$$\alpha : \{1, \ldots, n\} \to \{a_1, \ldots, a_n\}.$$

Similarly, the scheduling of network B is represented by a bijection

$$\beta : \{1, \ldots, n\} \to \{b_1, \ldots, b_n\}.$$

The number of wins of network A is given by

$$\#(\alpha, \beta) = |\{i \in n; \rho(\alpha(i)) > \rho(\beta(i))\}|.$$

We now choose the rating where $\rho(a_i) = i$ and $\rho(b_i) = i + \frac{1}{2}$ (for all $i = 1, \ldots, n$).

We claim that every pair (α, β) of schedules is unstable, i.e., at least one of the following two statements is true:

(A) Network A can change α so that $\#(\alpha, \beta)$ increases.

(B) Network B can change β so that $\#(\alpha, \beta)$ decreases.

Here is how: Note that $\#(\alpha, \beta)$ lies between 0 and $n-1$. Indeed, $\#(\alpha, \beta) = 0$ if (and only if) for all $i = 1, \ldots, n$, we have a_i and b_i in the same slot. Also, $\#(\alpha, \beta) = n - 1$ if (and only if) for all $i = 1, \ldots, n-1$, we have a_{i+1} and b_i in the same slot (this implies a_1 and b_n are in the same slot). Now notice that:

(A') If $\#(\alpha, \beta) < n - 1$, then network A can surely reschedule α so that $\#(\alpha, \beta) = n - 1$. Thus network A has improved unilaterally.

(B') If $\#(\alpha, \beta) > 0$, then network B can surely reschedule β so that $\#(\alpha, \beta) = 0$. Thus network B has improved unilaterally.

Since either (A') or (B') must be true, we are done.

4 Question 4

(5 points) Consider the following statement:

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{R})(\exists z \in \mathbb{R}) \left[(x > 0) \Rightarrow ((y^2 < x < y) \wedge (z < x < z^2) \wedge (z < y))\right] \quad (1)$$

Note that the range of variable x is $\mathbb{Z}$, not $\mathbb{R}$. This is called a "universal statement" because the leading quantifier is the universal quantifier ($\forall$). Similarly, we have "existential statements".

(i) Negate the statement (1), and then apply De Morgan's law to rewrite the result as an existential statement.

HINT: all our formal statements have the form $(Q_1)(Q_2)\cdots(Q_n)\,[\ldots predicate \ldots]$ where Q_i is the ith quantifier part. A predicate is a function from some domain to the set of Boolean values (true or false). E.g., the predicate "x is married" might be written as the function $Married(x)$ whose domain is the set of humans; $Married(x)$ is true iff x is married.

(ii) Is the statement (1) true? Justify it or give a counter example.

SOLUTION:

(i)

$$(\exists x \in \mathbb{Z})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R}) \left[(x > 0) \wedge (\neg(y^2 < x < y) \vee \neg(z < x < z^2) \vee (z \geq y))\right]$$

(ii)

This is false. A counter example is $x = 1$. In order for $y^2 < x < y$, we must have $y < 1$ (so that $y^2 < y$). But then $x < y$ cannot be true.

5 Question 5

(5 points) We provide the following program MergeSort.java, and an accompanying Makefile for you. The MergeSort program is mostly written for you – you just need to fill in the "merge" routine. The purpose of this question is to (a) introduce you to the Makefile program, (b) ensure that you have a basic Java programming environment.

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