MACHINE LEARNING AND PATTERN RECOGNITION

Spring 2004, Lecture 6: Gradient-Based Learning III: Architectures

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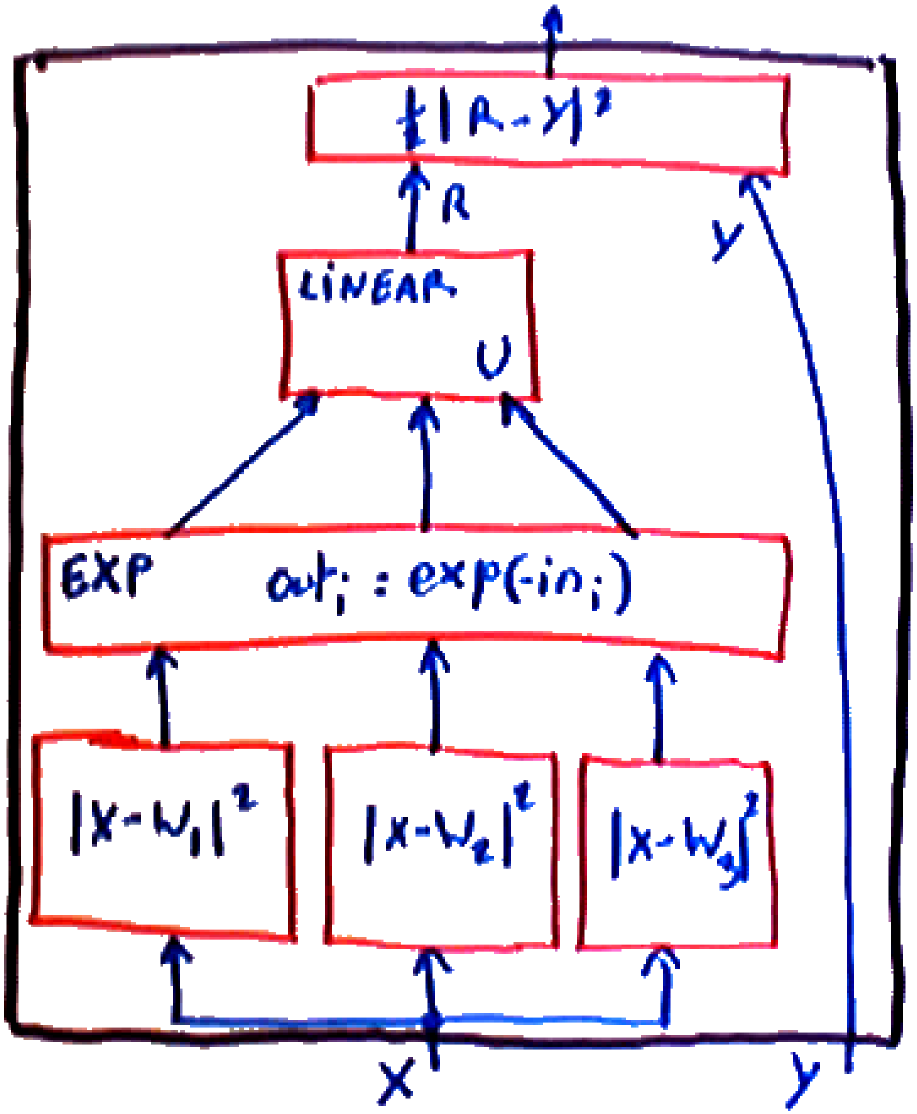
Radial Basis Function Network (RBF Net)



- Linearly combined Gaussian bumps.
- $F(X, W, U) = \sum_i u_i \exp(-k_i(X - W_i)^2)$
- The centers of the bumps can be initialized with the K-means algorithm (see below), and subsequently adjusted with gradient descent.
- This is a good architecture for regression and function approximation.

MAP/MLE Loss and Cross-Entropy

- classification (y is scalar and discrete). Let's denote $E(y, X, W) = E_y(X, W)$
- MAP/MLE Loss Function:

$$L(W) = \frac{1}{P} \sum_{i=1}^{P} [E_{y^i}(X^i, W) + \frac{1}{\beta} \log \sum_k \exp(-\beta E_k(X^i, W))]$$

- This loss can be written as

$$L(W) = \frac{1}{P} \sum_{i=1}^{P} -\frac{1}{\beta} \log \frac{\exp(-\beta E_{y^i}(X^i, W))}{\sum_k \exp(-\beta E_k(X^i, W))}$$

Cross-Entropy and KL-Divergence

- let's denote $P(j|X^i, W) = \frac{\exp(-\beta E_j(X^i,W))}{\sum_k \exp(-\beta E_k(X^i,W))}$, then

$$L(W) = \frac{1}{P}\sum_{i=1}^{P}\frac{1}{\beta}\log\frac{1}{P(y^i|X^i, W)}$$

$$L(W) = \frac{1}{P}\sum_{i=1}^{P}\frac{1}{\beta}\sum_{k} D_k(y^i)\log\frac{D_k(y^i)}{P(k|X^i, W)}$$

  with $D_k(y^i) = 1$ iff $k = y^i$, and 0 otherwise.

- example1: $D = (0, 0, 1, 0)$ and $P(.|X_i, W) = (0.1, 0.1, 0.7, 0.1)$. with $\beta = 1$, $L^i(W) = \log(1/0.7) = 0.3567$
- example2: $D = (0, 0, 1, 0)$ and $P(.|X_i, W) = (0, 0, 1, 0)$. with $\beta = 1$, $L^i(W) = \log(1/1) = 0$

Cross-Entropy and KL-Divergence

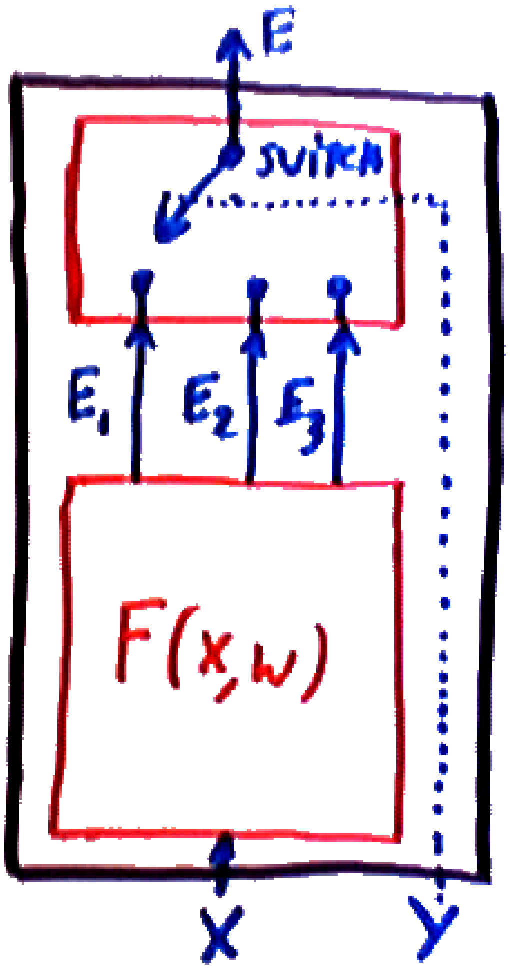
$$L(W) = \frac{1}{P} \sum_{i=1}^{P} \frac{1}{\beta} \sum_k D_k(y^i) \log \frac{D_k(y^i)}{P(k|X^i, W)}$$

- $L(W)$ is proportional to the *cross-entropy* between the conditional distribution of y given by the machine $P(k|X^i, W)$ and the *desired* distribution over classes for sample i, $D_k(y^i)$ (equal to 1 for the desired class, and 0 for the other classes).
- The cross-entropy also called *Kullback-Leibler divergence* between two distributions $Q(k)$ and $P(k)$ is defined as:

$$\sum_k Q(k) \log \frac{Q(k)}{P(k)}$$

- It measures a sort of dissimilarity between two distributions.
- the KL-divergence is not a distance, because it is not symmetric, and it does not satisfy the triangular inequality.

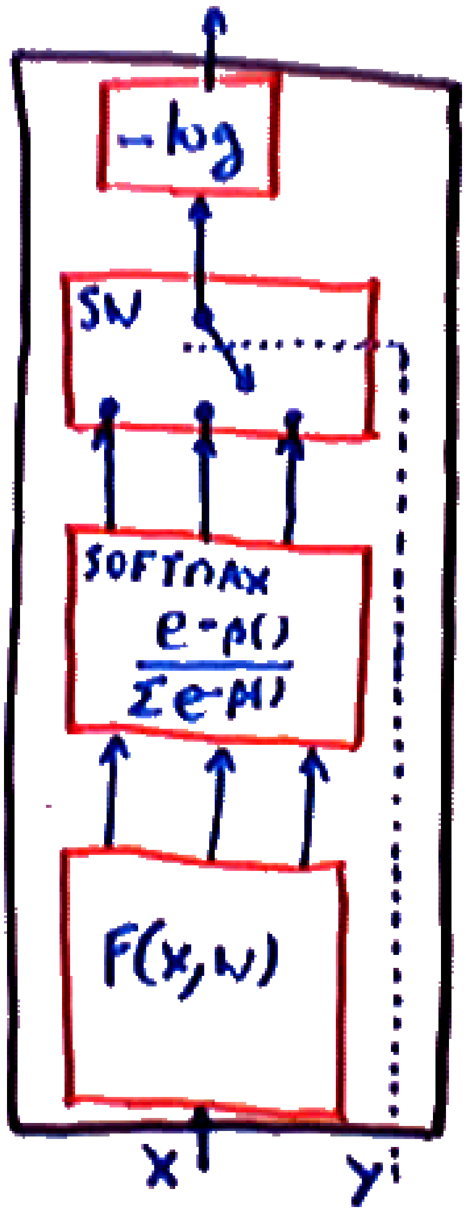
Multiclass Classification and KL-Divergence



- Assume that our discriminant module $F(X, W)$ produces a vector of energies, with one energy $E_k(X, W)$ for each class.
- A switch module selects the smallest E_k to perform the classification.
- As shown above, the MAP/MLE loss below be seen as a KL-divergence between the desired distribution for y, and the distribution produced by the machine.

$$L(W) = \frac{1}{P}\sum_{i=1}^{P}[E_{y^i}(X^i, W) + \frac{1}{\beta}\log\sum_k \exp(-\beta E_k(X^i, W))]$$

Multiclass Classification and Softmax



- The previous machine: discriminant function with one output per class + switch, with MAP/MLE loss
- It is equivalent to the following machine: discriminant function with one output per class + softmax + switch + log loss

$$L(W) = \frac{1}{P}\sum_{i=1}^{P}\frac{1}{\beta} - \log P(y^i|X, W)$$

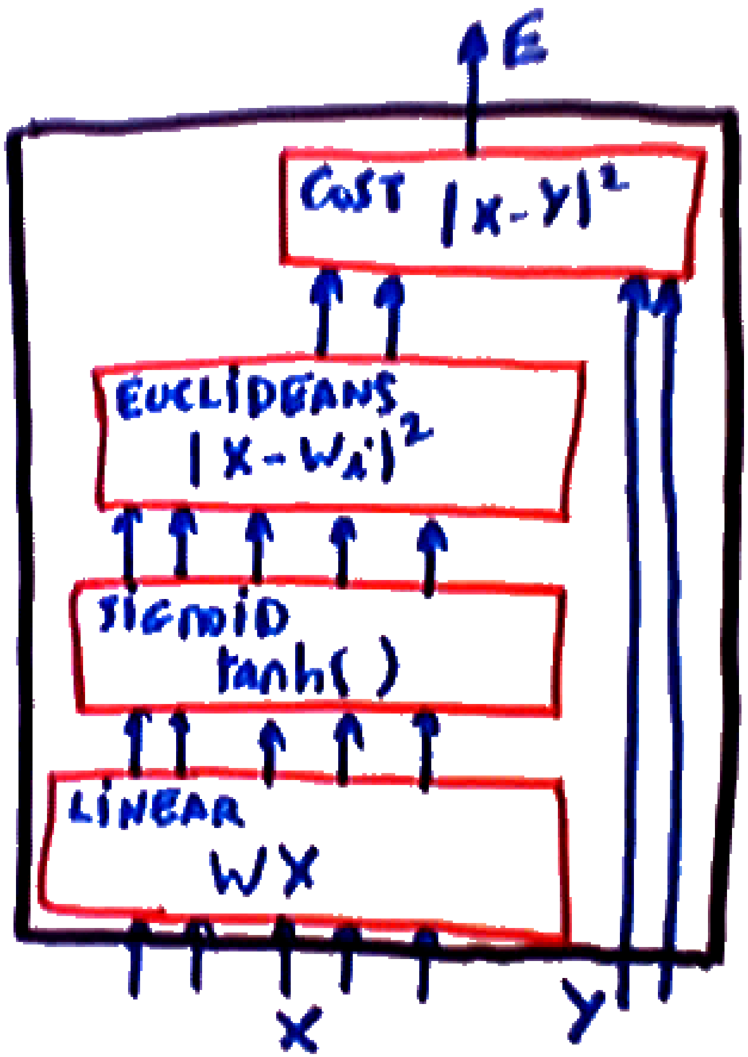
with $P(j|X^i, W) = \frac{\exp(-\beta E_j(X^i,W))}{\sum_k \exp(-\beta E_k(X^i,W))}$ (softmax of the $-E_j$'s).

- Machines can be transformed into various equivalent forms to factorize the computation in advantageous ways.

Multiclass Classification with a Junk Category

- Sometimes, one of the categories is “none of the above”, how can we handle that?
- We add an extra energy wire E_0 for the “junk” category which does not depend on the input. E_0 can be a hand-chosen constant or can be equal to a trainable parameter (let’s call it w_0).
- everything else is the same.

NN-RBF Hybrids

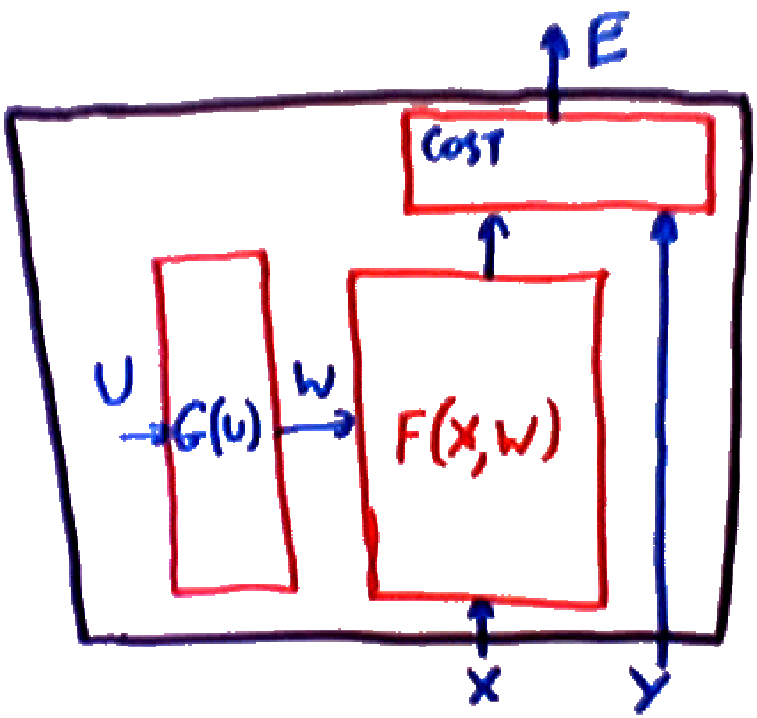


- sigmoid units are generally more appropriate for low-level feature extraction.
- Euclidean/RBF units are generally more appropriate for final classifications, particularly if there are many classes.
- Hybrid architecture for multiclass classification: sigmoids below, RBFs on top + softmax + log loss.

Parameter-Space Transforms

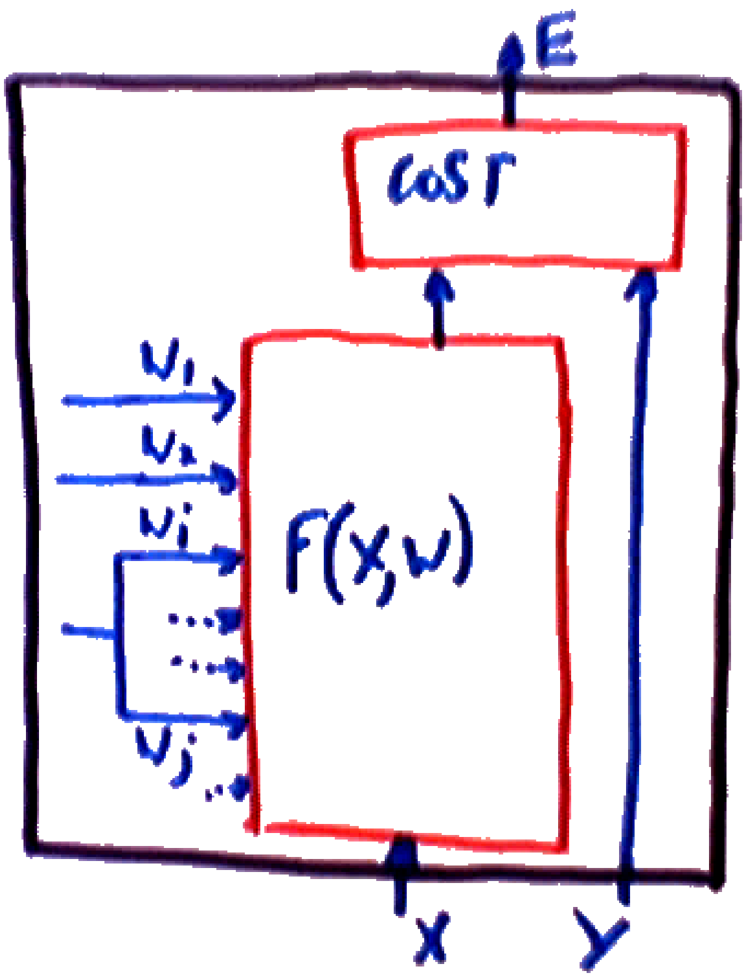
Reparameterizing the function by transforming the space

$$E(Y, X, W) \rightarrow E(Y, X, G(U))$$



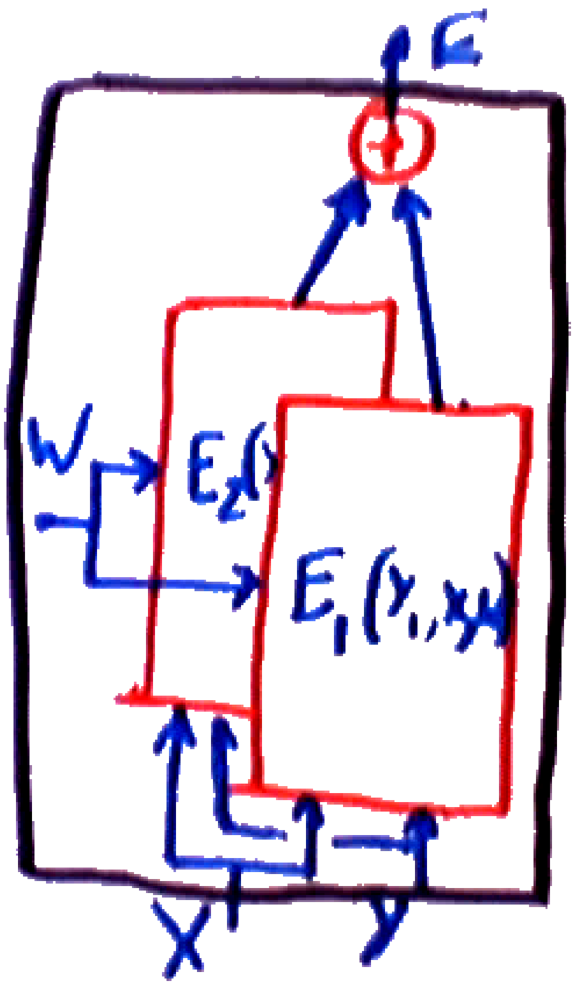
- gradient descent in U space:
  $U \leftarrow U - \eta \frac{\partial G}{\partial U}' \frac{\partial E(Y,X,W)}{\partial W}'$
- equivalent to the following algorithm in W space: $W \leftarrow W - \eta \frac{\partial G}{\partial U} \frac{\partial G}{\partial U}' \frac{\partial E(Y,X,W)}{\partial W}'$
- dimensions: $[N_w \times N_u][N_u \times N_w][N_w]$

Parameter-Space Transforms: Weight Sharing



- A single parameter is replicated multiple times in a machine
- $E(Y, X, w_1, \ldots, w_i, \ldots, w_j, \ldots) \rightarrow E(Y, X, w_1, \ldots, u_k, \ldots, u_k, \ldots)$
- gradient: $\frac{\partial E()}{\partial u_k} = \frac{\partial E()}{\partial w_i} + \frac{\partial E()}{\partial w_j}$
- w_i and w_j are tied, or equivalently, u_k is shared between two locations.

Parameter Sharing between Replicas



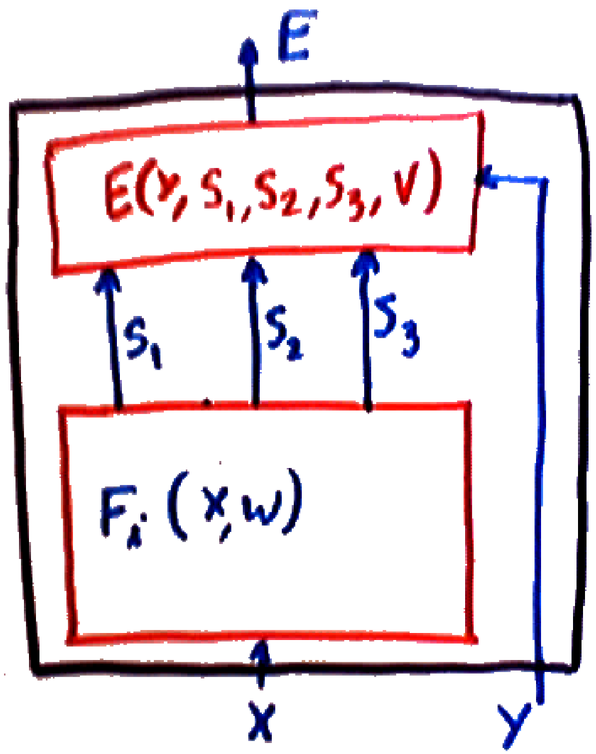
- We have seen this before: a parameter controls several replicas of a machine.
-

$$E(Y_1, Y_2, X, W) = E_1(Y_1, X, W) + E_1(Y_2, X, W)$$

- gradient:
$$\frac{\partial E(Y_1,Y_2,X,W)}{\partial W} = \frac{\partial E_1(Y_1,X,W)}{\partial W} + \frac{\partial E_1(Y_2,X,W)}{\partial W}$$
- W is shared between two (or more) instances of the machine: just sum up the gradient contributions from each instance.

Path Summation (Path Integral)

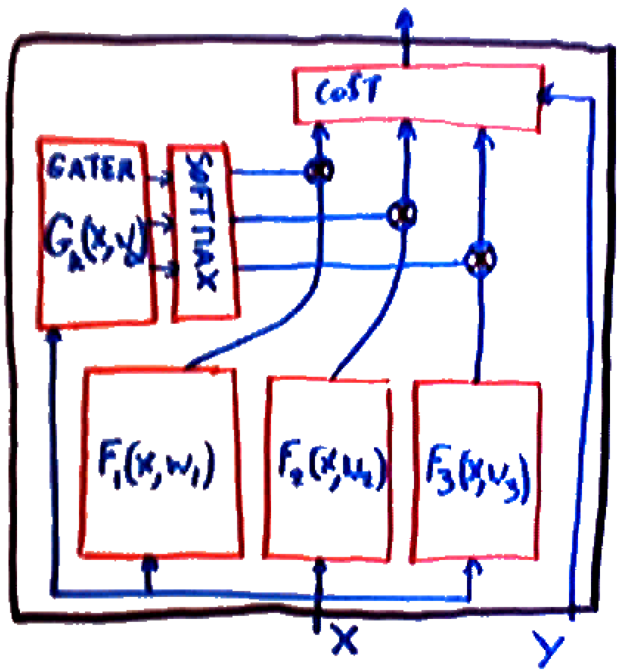
One variable influences the output through several others



- $E(Y, X, W) =$
  $E(Y, F_1(X, W), F_2(X, W), F_3(X, W), V)$
- gradient: $\frac{\partial E(Y,X,W)}{\partial X} = \sum_i \frac{\partial E_i(Y,S_i,V)}{\partial S_i} \frac{\partial F_i(X,W)}{\partial X}$
- gradient: $\frac{\partial E(Y,X,W)}{\partial W} = \sum_i \frac{\partial E_i(Y,S_i,V)}{\partial S_i} \frac{\partial F_i(X,W)}{\partial W}$
- there is no need to implement these rules explicitely. They come out naturally of the object-oriented implementation.

Mixtures of Experts

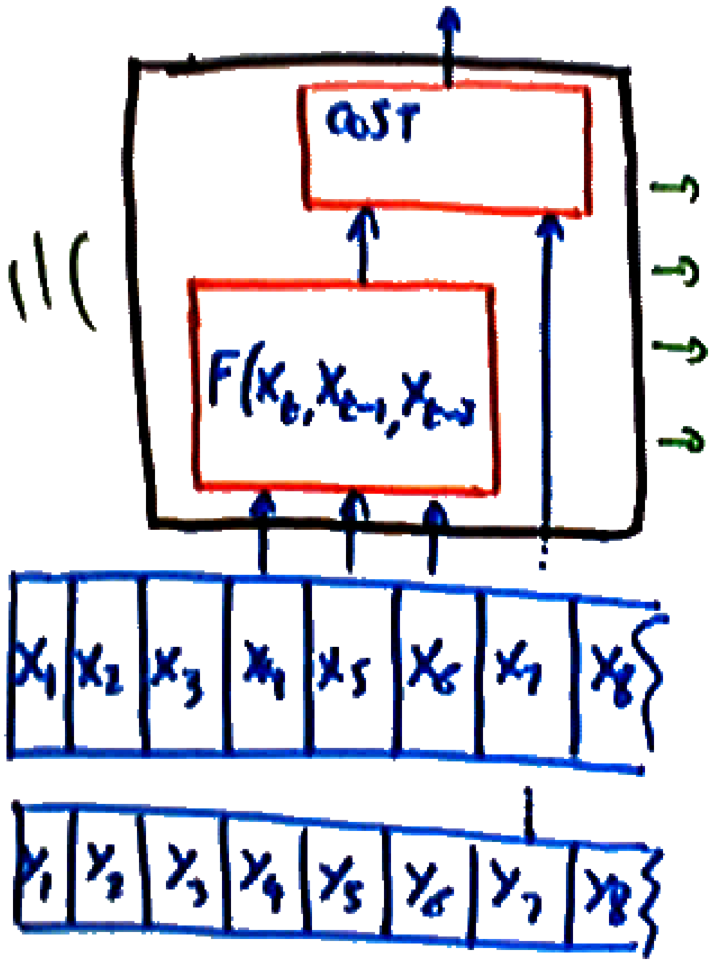
Sometimes, the function to be learned is consistent in restricted domains of the input space, but globally inconsistent. Example: piecewise linearly separable function.



- Solution: a machine composed of several "experts" that are specialized on subdomains of the input space.
- The output is a weighted combination of the outputs of each expert. The weights are produced by a "gater" network that identifies which subdomain the input vector is in.
- $F(X,W) = \sum_k u_k F^k(X,W^k)$ with $u_k = \frac{\exp(-\beta G_k(X,W^0))}{\sum_k \exp(-\beta G_k(X,W^0))}$
- the expert weights u_k are obtained by softmax-ing the outputs of the gater.
- example: the two experts are linear regressors, the gater is a logistic regressor.

Sequence Processing: Time-Delayed Inputs

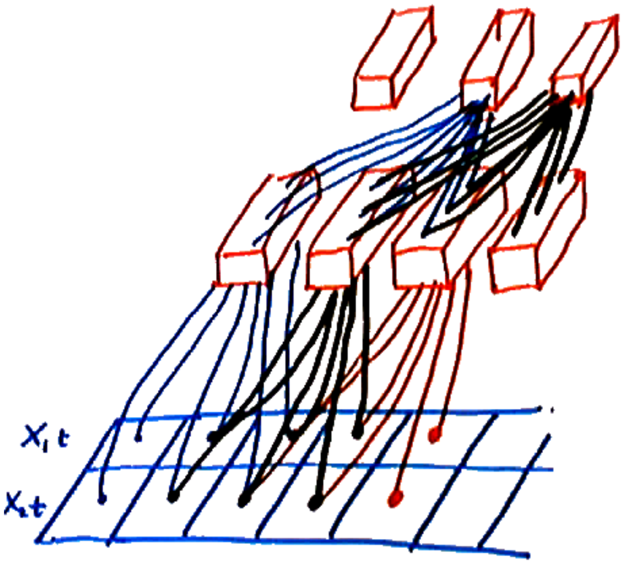
The input is a sequence of vectors X_t.



- simple idea: the machine takes a time window as input
- $R = F(X_t, X_{t-1}, X_{t-2}, W)$
- Examples of use:
  - predict the next sample in a time series (e.g. stock market, water consumption)
  - predict the next character or word in a text
  - classify an intron/exon transition in a DNA sequence

Sequence Processing: Time-Delay Networks

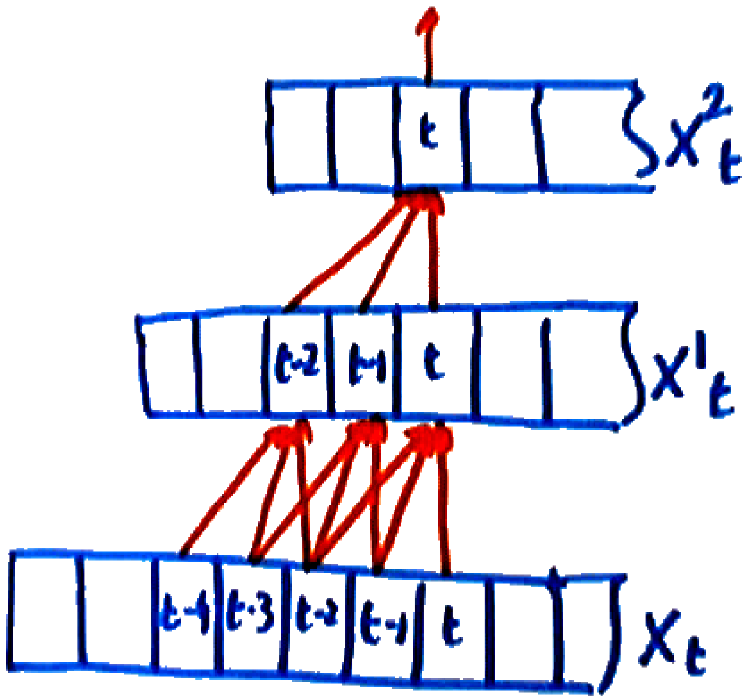
One layer produces a sequence for the next layer: stacked time-delayed layers.



- layer1 $X_t^1 = F^1(X_t, X_{t-1}, X_{t-2}, W^1)$
  layer2 $X_t^2 = F^1(X_t^1, X_{t-1}^1, X_{t-2}^1, W^2)$
  cost $E_t = C(X_t^1, Y_t)$
- Examples:
  - predict the next sample in a time series with long-term memory (e.g. stock market, water consumption)
  - recognize spoken words
  - recognize gestures and handwritten characters on a pen computer.
- How do we train?

Training a TDNN

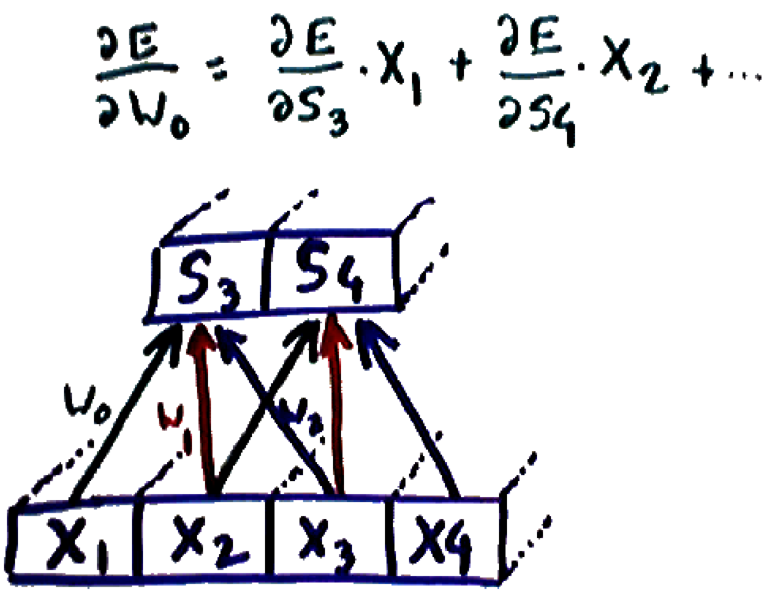
Idea: isolate the minimal network that influences the energy at one particular time step t.



- in our example, this is influenced by 5 time steps on the input.
- train this network in isolation, taking those 5 time steps as the input.
- Surprise: we have three identical replicas of the first layer units that share the same weights.
- We know how to deal with that.
- do the regular backprop, and add up the contributions to the gradient from the 3 replicas

Convolutional Module

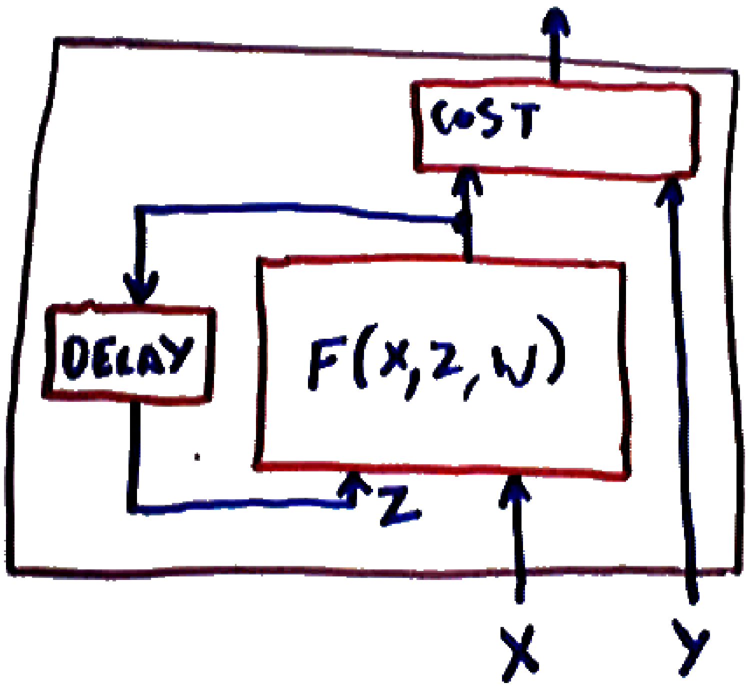
If the first layer is a set of linear units with sigmoids, we can view it as performing a sort of *multiple discrete convolutions* of the input sequence.



- 1D convolution operation: $S_t^1 = \sum_{j=1}^{T} W_j^{1\prime} X_{t-j}$.
- $w_j k \;\; j \in [1, T]$ is a *convolution kernel*
- sigmoid $X_t^1 = \tanh(S_t^1)$
- derivative: $\frac{\partial E}{\partial w_j^1 k} = \sum_{t=1}^{3} \frac{\partial E}{\partial S_t^1} X_{t-j}$

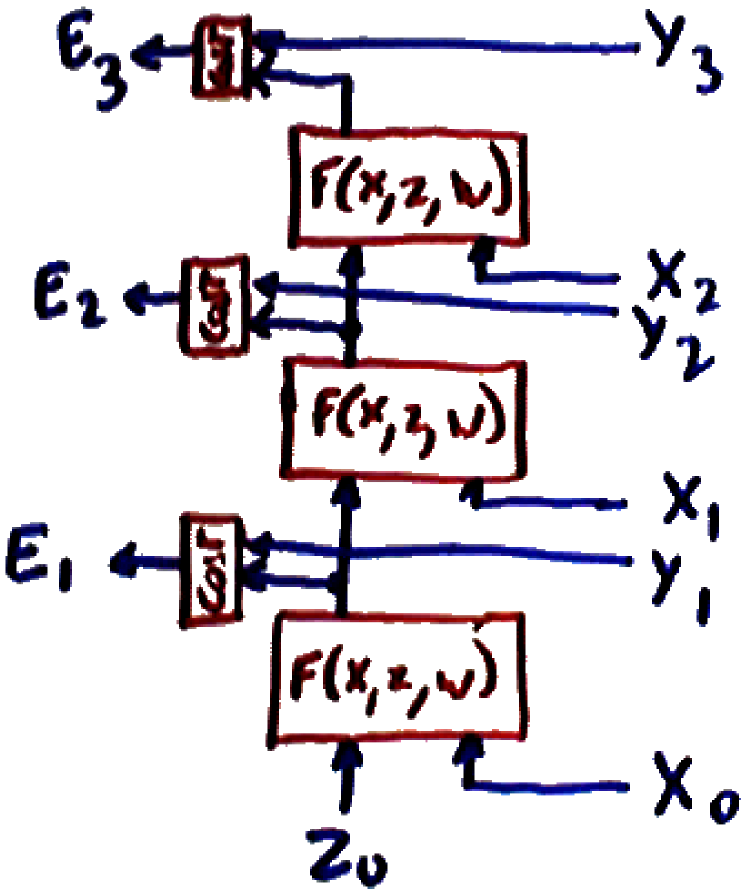
Simple Recurrent Machines

The output of a machine is fed back to some of its inputs Z. $Z_{t+1} = F(X_t, Z_t, W)$, where t is a time index. The input X is not just a vector but a sequence of vectors X_t.



- This machine is a *dynamical system* with an internal state Z_t.
- Hidden Markov Models are a special case of recurrent machines where F is linear.

Unfolded Recurrent Nets and Backprop through time



- To train a recurrent net: “unfold” it in time and turn it into a feed-forward net with as many layers as there are time steps in the input sequence.
- An unfolded recurrent net is a very “deep” machine where all the layers are identical and share the same weights.
- $\frac{\partial E}{\partial W} = \sum_t \frac{\partial E}{\partial Z_t} \frac{\partial F(X_t, Z_t, W)}{\partial W}$
- This method is called *back-propagation through time*.
- examples of use: process control (steel mill, chemical plant, pollution control....), robot control, dynamical system modelling...