Random Graphs G22.3033-007
Assignment 3. Due Monday, February 13,2006

1. Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$. Set $X = X_1 + \ldots + X_n$. Find $E[X^2]$ precisely. Find $E[X^4]$ precisely. [Idea: Expand and use linearity of expectation.]

2. Use the alteration method as discussed in class to find a lower bound to the Ramsey Number $R(3,t)$. [To give you a head start, let the Red probability be $\epsilon n^{-2/3}$ for some small ϵ, where n is the number of vertices.]

3. (a) Find an asymptotic formula for

$$\sum_{k=n^{1/2}}^{2n^{1/2}} (n)_k n^{-k}$$

by parametrizing $k = cn^{1/2}$ and turning it into an integral which can be evaluated numerically.

   (b) (*) Find an asymptotic formula for

$$\sum_{k=1}^{n} (n)_k n^{-k}$$

(Idea: Show that the sum over $k \le \epsilon n^{1/2}$ and over $k \ge Kn^{1/2}$ are (relatively) negligible for ϵ, K small and large respectively. The sum over $\epsilon n^{1/2} \le k \le Kn^{1/2}$ becomes an integral. As $\epsilon \to 0$ and $K \to \infty$ and becomes an integral from 0 to ∞ which has a nice closed form.)

4. Prove, for $m = m(n)$ as large as you can, the existence of an $n \times n$ matrix A of zeroes and ones with m ones which does not contain a 3×3 submatrix of all ones. Use the alteration method: make each entry one with probability p and then for each such submatrix change a one to zero. When you optimize [using Calculus!] your final answer should be of the form $m \sim an^b$ for some reasonable a, b.

5. Set $X = \sum_{i=1}^{n} X_i$ where $X_i = \pm 1$ uniformly and independently. Bound $\Pr[X > \frac{n}{2}]$ as follows.

   (a) Find a closed form for $E[e^{\lambda X_i}]$.

   (b) Find a closed form for $E[e^{\lambda X}]$.

   (c) Use the Chernoff Bound $\Pr[X > a] < E[e^{\lambda X}]e^{-\lambda a}$ with $a = \frac{n}{2}$. Use Calculus (this gets a little messy to put in closed form, full points for numerical answers) to select the the optimal λ.

   (d) Compare this with the lower bound

$$\Pr[X \geq \frac{n}{2}] \geq \Pr[X = \frac{n}{2}] = 2^{-n}\binom{n}{\frac{3n}{4}}$$

   showing that the upper and lower bounds have the same main terms.

It surprised him that he didn't dance very well. He had danced a lot in Connecticut, rather than make conversation, yet his finesse had flattened along one of those hyperbolic curves that computers delight in projecting. Men had been wrong ever to imagine the universe as a set of circles; in reality, nothing closes, everything approaches, but never quite touches, its asymptote.
from *I Will Not Let Thee Go, Except Thou Bless Me* by John Updike