Random Graphs G22.3033-007
Assignment 1. Due Monday, January 30,2006

1. The Bipartite Ramsey Number $BR(k)$ is the least n so that if A, B are disjoint with $|A| = |B| = n$ and $A \times B$ is two colored there exist $A_1 \subseteq A, B_1 \subseteq B$ with $|A_1| = |B_1| = k$ and $A_1 \times B_1$ monochromatic. Find and prove a theorem which gives a lower bound for $BR(k)$ and explore the asymptotics.

2. Let $f(k)$ be the maximal n for which there exists p with $0 \leq p \leq 1$ such that
$$n^k p^{k^2/2} + n^{2k}(1-p)^{2k^2} \leq 1$$
Let $U(k)$ be the maximal n for which there exists such p with $n^k p^{k^2/2} \leq 1$ and $n^{2k}(1-p)^{2k^2} \leq 1$. Let $L(k)$ be the maximal n for which there exists such p with $n^k p^{k^2/2} \leq \frac{1}{2}$ and $n^{2k}(1-p)^{2k^2} \leq \frac{1}{2}$.
   (a) Argue that $L(k) \leq f(k) \leq U(k)$
   (b) Find the asymptotics of $U(k)$. (Warning: Do *not* assume $p = o(1)$ because the optimal p isn't!) Partial credit for $\lim_k U(k)^{1/k}$.
   (c) Find the asymptotics of $L(k)$, showing that it is the same as that of $U(k)$. (That is, changing 1 to $\frac{1}{2}$ had an asymptotically negligible effect.)
   (d) Deduce the asymptotics of $f(k)$

3. Find asymptotic lower bounds on the Ramsey function $R(k, 2k)$. That is, set $g(k)$ to be the maximal n for which there exists p with $0 \leq p \leq 1$ such that
$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{2k}(1-p)^{\binom{2k}{2}} < 1$$
Find an asymptotic formula for $g(k)$. (Note: You'll want to use the ideas of the previous problem. Still, this is not an easy problem. Full marks for $\lim_k g(k)^{1/k}$ – its the same as $\lim_k f(k)^{1/k}$ but you have to prove this. The full asymptotics are if you enjoy a challenge.)

4. Find $m = m(n)$ as large as you can so that the following holds: Let $A_1, \ldots, A_m \subseteq \{1, \ldots, 4n\}$ with all $|A_i| = n$. Then there exists a two coloring of $\{1, \ldots, 4n\}$ such that none of the A_i are monochromatic. Use a random equicoloring of $\{1, \ldots, 4n\}$. Express your answer as an asymptotic function of n.