Let π be a random permutation of $\{1, 2 \leq n\}$, chosen according to a uniform distribution among all possible $n!$ permutations. Denote by T the set of all ordered fourtuples (i, j, i', j') satisfying $i < i', j \neq j'$ and $a_{ij} = a_{i'j'}$. For each $(i, j, i', j') \in T$, let $A_{iji'j'}$ denote the event that $\pi(i) = j$ and $\pi(i') = j'$. The existence of a Latin transversal is equivalent to the statement that with positive probability none of these events hold. Let us define a symmetric digraph, (i.e., a graph) G on the vertex set T by making (i, j, i', j') adjacent to (p, q, p', q') if and only if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\}$ $\cap\{q, q'\} \neq \emptyset$. Thus, these two fourtuples are not adjacent iff the four cells $(i, j), (i', j'), (p, q)$ and (p', q') occupy four distinct rows and columns of A. The maximum degree of G is less than $4nk$; indeed, for a given $(i, j, i', j') \in T$ there are $4n$ choices of (p, q) with either $p \in \{i, i'\}$ or $q \in \{j, j'\}$, and for each of these choices of (p, q) there are less than k choices for $(p', q') \neq (p, q)$ with $a_{pq} = a_{p'q'}$. Since $e \cdot 4nk \cdot \frac{1}{n(n-1)} \leq 1$, the desired result follows from the above mentioned strengthening of the symmetric version of the Lovász Local Lemma, if we can show that

$$\Pr(A_{iji'j'} | \bigwedge_S \overline{A_{pqp'q'}}) \leq 1/n(n-1)$$

for any $(i, j, i', j') \in T$ and any set S of members of T which are nonadjacent in G to (i, j, i', j'). By symmetry, we may assume that $i = j = 1, i' = j' = 2$ and that hence none of the p's nor q's are either 1 or 2. Let us call a permutation π *good* if it satisfies $\bigwedge_S \overline{A_{pqp'q'}}$, and let S_{ij} denote the set of all good permutations π satisfying $\pi(1) = i$ and $\pi(2) = j$. We claim that $|S_{12}| \leq |S_{ij}|$ for alll $i \neq j$. Indeed, suppose first that $i, j > 2$. For each good $\pi \in S_{12}$ define a permutation π^* as follows. Suppose $\pi(x) = i$, $\pi(y) = j$. Then define $\pi^*(1) = i, \pi^*(2) = j, \pi^*(x) = 1, \pi^*(y) = 2$ and $\pi^*(t) = \pi(t)$ for all $t \neq 1, 2, x, y$. One can easily check that π^* is good, since the cells $(1, i), (2, j), (x, 1), (y, 2)$ are not part of any $(p, q, p', q') \in S$. Thus $\pi^* \in S_{ij}$, and since the mapping $\pi \rightarrow \pi^*$ is injective $|S_{12}| \leq |S_{ij}|$, as claimed. Similarly one can define injective mappings showing that $|S_{12}| \leq |S_{ij}|$ even when $\{i, j\} \cap \{1, 2\} \neq \emptyset$. It follows that $\Pr(A_{1122} \bigwedge \bigwedge_S \overline{A_{pqp'q'}}) \leq Pr(A_{1i2j} \bigwedge \bigwedge_S \overline{A_{pqp'q'}})$ for all $i \neq j$ and hence that $Pr(A_{1122} | \bigwedge \bigwedge_S \overline{A_{pqp'q'}}) \leq 1/n(n-1)$. By symmetry, this implies (5.1) and completes the proof.□

complete graph on S is red. Clearly $Pr(A_T) = p^3$ and $Pr(B_S) = (1-p)^{\binom{k}{2}}$. Construct a dependency digraph for the events A_T and B_S by joining two vertices by edges (in both directions) iff the corresponding complete graphs share an edge. Clearly, each A_T-node of the dependency graph is adjacent to $3(n-3) < 3n$ $A_{T'}$-nodes and to at most $\binom{n}{k}$ $B_{S'}$-nodes. Similarly, each B_S-node is adjacent to $\binom{k}{2}(n-k) < k^2n/2$ A_T nodes and to at most $\binom{n}{k}$ $B_{S'}$-nodes. It follows from the general case of the Lovász Local Lemma that if we can find a $0 < p < 1$ and two real numbers $0 \le x < 1$ and $0 \le y < 1$ such that

$$p^3 \le x(1-x)^{3n}(1-y)^{\binom{n}{k}}$$

and

$$(1-p)^{\binom{k}{2}} \le y(1-x)^{k^2n/2}(1-y)^{\binom{n}{k}}$$

then $R(k,3) > n$.

Our objective is to find the largest possible $k = k(n)$ for which there is such a choice of p, x and y. An elementary computation (if you have a spare weekend!) shows that the best choice is when $p = c_1 n^{-1/2}$, $k = c_2 n^{1/2} \log n, x = c_3/n^{3/2}$ and $y = \frac{c_4}{e^{n^{1/2}\log^2 n}}$. This gives that $R(k,3) > c_5 k^2/\log^2 k$. A similar argument gives that $R(k,4) > k^{5/2+o(1)}$. In both cases the amount of computation required is considerable. However, the hard work does pay; the bound $R(k,3) > c_5 k^2/\log^2 k$ matches a lower bound of Erdős proved in 1961 by a highly complicated probabilistic argument. The bound above for $R(k,4)$ is better than any bound for $R(k,4)$ known to be proven without the Local Lemma.

4 Latin Transversals

Following the proof of the Lovász Local Lemma we noted that the mutual independency assumption in this lemma can be replaced by the weaker assumption that the conditional probability of each event, given the mutual non-occurance of an arbitrary set of events, each nonadjacent to it in the dependency digraph, is sufficiently small. In this section we describe an application, from Erdős-Spencer [1991], of this modified version of the lemma. Let $A = (a_{ij})$ be an n of n matrix with, say, integer entries. A permutation π is called a *Latin transversal* (of A) if the entries $a_{i\pi(i)}$ $(1 \le i \le n)$ are all distinct.

Theorem 5.1. Suppose $k \le (n-1)/(4e)$ and suppose that no integer appears in more than k entries of A. Then A has a Latin Transversal.

probability space none of them holds. In fact, there are trivial examples of countably many mutually independent events A_i, satisfying $Pr(A_i) = 1/2$ and $\bigwedge_{i\geq 1} \overline{A_i} = \emptyset$. Thus the compactness argument is essential in the above proof.

3 Lower bounds for Ramsey numbers

The deriviation of lower bounds for Ramsey numbers by Erdős in 1947 was one of the first applications of the probabilistic method. The Lovász Local Lemma provides a simple way of improving these bounds. Let us obtain, first, a lower bound for the diagonal Ramsey number $R(k, k)$. Consider a random 2-coloring of the edges of K_n. For each set S of k vertices of K_n, let A_S be the event that the complete graph on S is monochromatic. Clearly $\Pr(A_S) = 2^{1-\binom{k}{2}}$. It is obvious that each event A_S is mutually independent of all the events A_T, but those which satisfy $|S \cap T| \geq 2$, since this is the only case in which the corresponding complete graphs share an edge. We can therefore apply Corollary 1.2 with $p = 2^{1-\binom{k}{2}}$ and $d = \binom{k}{2}\binom{n}{k-2}$ to conclude; Proposition 3.1. If $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) \cdot 2^{1-\binom{k}{2}} < 1$ then $R(k, k) > n$.

A short computation shows that this gives $R(k, k) > \frac{\sqrt{2}}{e}(1 + o(1))k2^{k/2}$, only a factor 2 improvement on the bound obtained by the straightforward probabilistic method. Although this minor improvement is somewhat disappointing it is certainly not surprising; the Local Lemma is most powerful when the dependencies between events are rare, and this is not the case here. Indeed, there is a total number of $K = \binom{n}{k}$ events considered, and the maximum outdegree d in the dependency digraph is roughly $\binom{k}{2}\binom{n}{k-2}$. For large k and much larger n (which is the case of interest for us) we have $d > K^{1-O(1/k)}$, i.e., quite a lot of dependencies. On the other hand, if we consider small sets S, e.g., sets of size 3, we observe that out of the total $K = \binom{n}{3}$ of them each shares an edge with only $3(n-3) \approx K^{1/3}$. This suggests that the Lovász Local Lemma may be much more significant in improving the off-diagonal Ramsey numbers $R(k, l)$, especially if one of the parameters, say l, is small. Let us consider, for example, following Spencer (1977), the Ramsey number $R(k, 3)$. Here, of course, we have to apply the nonsymmetric form of the Lovász Local Lemma. Let us 2-color the edges of K_n randomly and independently, where each edge is colored blue with probability p. For each set of 3 vertices T, let A_T be the event that the triangle on T is blue. Similarly, for each set of k vertices S, let B_S be the event that the

hypergraph is has Property B. The next result we consider, which appeared in the original paper of Erdős and Lovász, deals with k-colorings of the real numbers. For a k-coloring $c : R \rightarrow \{1, 2 \ldots k\}$ of the real numbers by the k colors $1, 2 \ldots k$, and for a subset $T \subset R$, we say that T is *multicolored* (with respect to c) if $c(T) = \{1, 2 \ldots k\}$, i.e., if T contains elements of all colors.
Theorem 2.2. Let m and k be two positive integers satisfying

$$e\,(m(m-1)+1)\,k\left(1-\frac{1}{k}\right)^{m} \leq 1$$

Then, for any set S of m real numbers there is a k-coloring so that each translation $x + S$ (for $x \in R$) is multicolored.

Notice that the condition holds whenever $m > (3 + o(1))k \log k$. There is no known proof of existence of any $m = m(k)$ with this property without using the local lemma.

We first fix a *finite* subset $X \subset R$ and show the existence of a k-coloring so that each translation $x + S$ (for $x \in X$) is multicolored. This is an easy consequence of the Lovász Local Lemma. Indeed, put $Y = \bigcup_{x \in X}(x + S)$ and let $c : Y \rightarrow \{1, 2 \ldots k\}$ be a random k-coloring of Y obtained by choosing, for each $y \in Y$, randomly and independently, $c(y) \in \{1, 2 \ldots, k\}$ according to a uniform distribution on $\{1, 2 \ldots k\}$. For each $x \in X$, let A_x be the event that $x + S$ is not multicolored (with respect to c). Clearly $Pr(A_x) \leq k\left(1 - \frac{1}{k}\right)^m$. Moreover, each event A_x is mutually independent of all the other events $A_{x'}$ but those for which $(x + S) \cap (x' + S) \neq \emptyset$. As there are at most $m(m - 1)$ such events the desired result follows from Corollary 1.2.

We can now prove the existence of a coloring of the set of all reals with the desired properties, by a standard compactness argument. Since the discrete space with k points is (trivially) compact, Tychanov's Theorem (which is equivalent to the axiom of choice) implies that an arbitrary product of such spaces is compact. In particular, the space of all functions from the reals to $\{1, 2 \ldots k\}$, with the usual product topology, is compact. In this space for every fixed $x \in R$, the set C_x of all colorings c, such that $x+S$ is multicolored is closed. (In fact, it is both open and closed, since a basis to the open sets is the set of all colorings whose values are prescribed in a finite number of places). As we proved above, the intersection of any finite number of sets C_x is nonempty. It thus follows, by compactness, that the intersection of all sets C_x is nonempty. Any coloring in this intersection has the properties in the conclusion of Theorem 2.2.□

Note that it is impossible, in general, to apply the Lovász Local Lemma to an infinite number of events and conclude that in some point of the

If $d = 0$ the result is trivial. Otherwise, by the assumption there is a dependency digraph $D = (V, E)$ for the events $A_1 \dots A_n$ in which for each i $|\{j : (i,j) \in E\}| \le d$. The result now follows from Lemma 1.1 by taking $x_i = 1/(d+1)(< 1)$ for all i and using the fact that for any $d \ge 2$, $\left(1 - \frac{1}{d+1}\right)^d > 1/e$.□

It is worth noting that as shown by Shearer in 1985,the constant "e" is the best possible constant in inequality (1.5). Note also that the proof of Lemma 1.1 indicates that the conclusion remains true even when we replace the two assumptions that each A_i is mutually independent of $\{A_j : (i,j) \notin E)$ and that $Pr(A_i) \le x_i \prod_{(ij)\in E}(1 - x_j)$ by the weaker assumption that for each i and each $S_2 \subset \{1 \dots n\} - \{j : (i,j) \in E\}$, $Pr\left(x_i | \bigwedge_{j \in S_2} \overline{A_j}\right) \le x_i \prod_{(i,j)\in E}(1 - x_j)$. This turns out to be useful in certain applications.

In the next few sections we present various applications of the Lovász Local Lemma for obtaining combinatorial results. There is no known proof of any of these results, which does not use the this Lemma.

2 Property B and multicolored sets of real numbers

A hypergraph $H = (V, E)$ is said to have property B if there is a coloring of V by two colors so that no edge $f \in E$ is monochromatic.

Theorem 2.1. Let $H = (V, E)$ be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \le 2^{k-1}$ then H has property B.

Color each vertex v of H, randomly and independently, either blue or red (with equal probability). For each edge $f \in E$, let A_f be the event that f is monochromatic. Clearly $Pr(A_f) = 2/2^{|f|} \le 1/2^{k-1}$. Moreover, each event A_f is clearly mutually independent of all the other events $A_{f'}$ for all edges f' that do not intersect f. The result now follows from Corollary 1.2. □

A special case of Theorem 2.1 is that for any $k \ge 9$, any k-uniform k-regular hypergraph H has property B. Indeed, since any edge f of such an H contains k vertices, each of which is incident with k edges (including f), it follows that f intersects at most $d = k(k-1)$ other edges. The desired result follows, since $e(k(k-1)+1) < 2^{k-1}$ for each $k \ge 9$. This special case has a different proof (see [Alon-Bregman (1988)]), which works for each $k \ge 8$. It is plausible to conjecture that in fact for each $k \ge 4$ each k-uniform k-regular prove it for S. Put

$$S_1 = \{j \in S; (i,j) \in E\}, S_2 = S - S_1$$

Then

$$\Pr\left(A_i | \bigwedge_{j\in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \wedge (\bigwedge_{j\in S_1} \overline{A_j}) | \bigwedge_{l\in S_2} \overline{A_l}\right)}{\Pr\left(\bigwedge_{j\in S_1} \overline{A_j} | \bigwedge_{l\in S_2} \overline{A_l}\right)}$$

To bound the numerator observe that since A_i is mutually independent of the events $\{A_l : l \in S_2\}$

$$\Pr\left(A_i \wedge (\bigwedge_{j\in S_1} \overline{A_j}) | \bigwedge_{l\in S_2} \overline{A_l}\right) \le \Pr\left(A_i | \bigwedge_{l\in S_2} \overline{A_l}\right) = Pr(A_i) \le x_i \prod_{(i,j)\in E} (1-x_j)$$

The denominator, on the other hand, can be bounded by the induction hypothesis. Indeed, suppose $S_1 = \{j_1, j_2 \dots j_r\}$. If $r = 0$ then the denominator is 1, and (1.1) follows. Otherwise, setting $B = \wedge_{l\in S_2} \overline{A_l}$,

$$\Pr\left(\overline{A_{j_1}} \wedge \overline{A_{j_2}} \dots \overline{A_{j_r}} | B\right) = (1 - \Pr(A_{j_1}|B)) \cdot$$

$$\cdot \left(1 - \Pr\left(A_{j_2} | \overline{A_{j_1}} \wedge B\right)\right) \cdots \left(1 - \Pr\left(A_{j_r} | \overline{A_{j_1}} \wedge \dots \wedge \overline{A_{j_{r-1}}} \wedge B\right)\right)$$

$$\ge (1 - x_{j_1}) \cdots (1 - x_{j_r}) \ge \prod_{(i,j)\in E} (1 - x_j)$$

Substituting we conclude that $\Pr\left(A_i | \bigwedge_{j\in S} \overline{A_j}\right) \le x_i$, completing the proof of the induction.

The assertion of Lemma 1.1 now follows easily, as

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) = (1-Pr(A_1)) \cdot (1-Pr(A_2|\overline{A_1})) \cdot \ldots \cdot (1-Pr(A_n| \bigwedge_{i=1}^{n-1} \overline{A_i}) \ge \prod_{i=1}^{n} (1-x_i)$$

completing the proof. □

Corollary 1.2 (Lovász Local Lemma; Symmetric Case): Let $A_1, A_2 \dots A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $Pr(A_i) \le p$ for all $1 \le i \le n$. If

$$ep(d+1) \le 1$$

then $\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0$.

Lecture 9: The Lovász Local Lemma

1 The Lemma

In a typical probabilistic proof of a combinatorial result, one usually has to show that the probability of a certain event is positive. However, many of these proofs actually give more and show that the probability of the event considered is not only positive but is large. In fact, most probabilistic proofs deal with events that hold with high probability, i.e., a probability that tends to 1 as the dimensions of the problem grow. On the other hand, there is a trivial case in which one can show that a certain event holds with positive, though very small, probability. Indeed, if we have n mutually independent events and each of them holds with probability at least $p > 0$, then the probability that all events hold simultaneously is at least p^n, which is positive, although it may be exponentially small in n.

It is natural to expect that the case of mutual independence can be generalized to that of rare dependencies, and provide a more general way of proving that certain events hold with positive, though small, proability. Such a generalization is, indeed, possible, and is stated in the following lemma, known as the Lovász Local Lemma. This simple lemma, first proved in [Erdős-Lovász (1975)] is an extremely powerful tool, as it supplies a way for dealing with rare events.

Lemma 1.1 (The Local Lemma; General Case):

Let $A_1, A_2 \ldots A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2 \ldots n\}$ is called a *dependency digraph* for the events $A_1 \ldots A_n$ if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j : (i,j) \notin E\}$. Suppose that $D = (V, E)$ is a dependency digraph for the above events and suppose there are real numbers $x_1 \ldots x_n$ such that $0 \leq x_i < 1$ and $Pr(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ for all $1 \leq i \leq n$. Then $Pr\left(\bigwedge_{i=1}^{n} \overline{A_j}\right) \geq \prod\limits_{i=1}^{n}(1 - x_i)$. In particular, with positive probability no event A_i holds.

We first prove, by induction on s, that for any $S \subset \{1 \ldots n\}$, $|S| = s < n$ and any $i \notin S$

$$\Pr\left(A_i | \bigwedge_{j\in S} \overline{A_j}\right) \leq x_i$$

This is certainly true for $s = 0$. Assuming it holds for all $s' < s$, we