Chapter 3: Alterations

The basic probabilistic method was described in Chapter 1 as follows: Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this space with positive probabilities. In this chapter we consider situations where the "random" structure does not have all the desired properties but may have a few "blemishes". With a small alteration we remove the blemishes giving the desired structure.

1 Ramsey Numbers

Recall from §1.1 that $R(k,l) > n$ means there exists a two-coloring of the edges of K_n by red and blue so that there is neither a red K_k nor a blue K_l. Theorem 1.1 For any integer n

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Proof. Consider a random two-coloring of the edges of K_n obtained by coloring each edge independently either red or blue, where each color is equally likely. For any set R of k vertices let X_R be the indicator random variable for the event that the induced subgraph of K_n on R is monochromatic. Set $X = \sum X_R$, the sum over all such R. ¿From Linearity of Expectation

$$E[X] = \sum E[X_R] = m \text{ with } m = \binom{n}{k} 2^{1-\binom{k}{2}}$$

Thus there exists a two-coloring for which $X \leq m$. Fix such a coloring. Remove from K_n one vertex from each monochromatic k-set. At most m vertices have been removed (we may have "removed" the same vertex more than once but this only helps) so s vertices remain with $s \geq n - m$. This coloring on these s points has no monochromatic k-set. □

We are left with the "calculus" problem of finding that n which will optimize the inequality. Some analysis shows that we should take $n \sim e^{-1}k2^{k/2}(1 - o(1))$ giving

$$R(k,k) > \frac{1}{e}(1 + o(1))k2^{k/2}$$

A careful examination of Theorem 1.1 of Chapter 1 gives the lower bound

$$R(k,k) > \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$$

The more powerful Lovasz Local Lemma - see Theorem 3.1 of Chapter 5- gives

$$R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$$

The distinctions between these bounds may be considered inconsequential since the best known upper bound for $R(k,k)$ is on the order $(4+o(1))^n$. We give all three lower bounds in following our philosophy of emphasizing *methodologies* rather than results. There is no known probabilistic proof of Ramsey's Theorem. For this reason we do not discuss proofs of upper bounds for the Ramsey function in this volume. *Ramsey Theory* (Graham, Rothschild, Spencer [1990]) covers these topics in detail.

In dealing with the off-diagonal Ramsey Numbers the disinction between the basic method and the alteration is given in the following two results.
Theorem 1.1 If there exists $p \in [0,1]$ with

$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)\binom{l}{2} < 1$$

then $R(k,l) > n$.
Theorem 1.3 For all integers n and $p \in [0,1]$

$$R(k,l) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{l}(1-p)\binom{l}{2}$$

Proofs. In both cases we consider a random two-coloring of K_n obtained by coloring each edge independently either red or blue, where each edge is red with probability p. Let X be the number of red k-sets plus the number of blue l-sets. Linearity of Expectation gives

$$E[X] = \binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)\binom{l}{2}$$

For Theorem 1.2 $E[X] < 1$ so there exists a two-coloring with $X = 0$. For Theorem 1.3 there exists a two-coloring with s "bad" sets (either red k-sets or blue l-sets), $s \leq E[X]$. Removing one point from each bad set gives a coloring of at least $n - s$ points with no bad sets. □□

The asymptotics of Theorems 1.2,1.3 can get fairly complex. Oftentimes Theorem 1.3 gives a substantial improvement on Theorem 2. Even further improvements may be found using the Lovasz Local Lemma. These bounds have been analyzed in Spencer [1977].

2 Independent Sets

Here is a short and sweet argument that gives roughly half of the celebrated Turan's Theorem. $\alpha(G)$ is the independence number of a graph G, $\alpha(G) \geq t$ means there exist t vertices with no edges between them.
Theorem 2.1 Let $G = (V, E)$ have v vertices and $nd/2$ edges, $d \geq 1$. Then $\alpha(G) \geq n/2d$.
Proof. Let $S \subseteq V$ be a random subset defined by

$$\Pr[v \in S] = p,$$

p to be determined, the events $v \in S$ being mutually independent. Let $X = |S|$ and let Y be the number of edges in $G|_S$. For each $e = \{i, j\} \in E$ let Y_e be the indicator random variable for the event $i, j \in S$ so that $Y = \sum_{e \in E} Y_e$. For any such e

$$E[Y_e] = \Pr[i, j \in S] = p^2$$

so by Linearity of Expectation

$$E[Y] = \sum_{e \in E} E[Y_e] = \frac{nd}{2}p^2$$

Clearly $E[X] = np$ so, again by Linearity of Expectation

$$E[X - Y] = np - \frac{nd}{2}p^2$$

We set $p = 1/d$ (here using $d \geq 1$) to maximize this quantity, giving

$$E[X - Y] = \frac{n}{2d}$$

Thus there exists a specific S for whom the number of vertices of S minus the number of edges in S is at least $n/2d$. Select one vertex from each edge of S and delete it. This leaves a set S^* with at least $n/2d$ vertices. All edges having been destroyed, S^* is an independent set. $\square$

3 Combinatorial Geometry

For a set S of n points in the unit square I, let $T(S)$ be the minimum area of a triangle whose vertices are three distinct points of S. Put $T(n) = \max T(S)$, where S ranges over all sets of n points in I. Heilbronn conjectured that $T(n) = O(1/n^2)$. This conjecture was disproved by Komlós, Pintz and Szemerédi (1982) who showed, by a rather involved probabilistic construction, that there is a set S of n points in I such that $T(S) = \Omega(\log n/n^2)$. As this argument is rather complicated, we only present here a simpler one showing that $T(n) = \Omega(1/n^2)$.
Theorem 3.1 There is a set S of n points in the unit square I such that $T(S) \geq 1/(100n^2)$.

We first make a calculation. Let P, Q, R be independently and uniformly selected from I and let $\mu = \mu(PQR)$ denote the area of the triangle PQR. We bound $\Pr[\mu \leq \epsilon]$ as follows. Let x be the distance from P to Q so that $\Pr[b \leq x \leq b + \Delta b] \leq \pi(b+\Delta b)^2 - \pi b^2$ and in the limit (using infinitesmals) $\Pr[b \leq x \leq b + db] \leq 2\pi b db$. Given P, Q at distance b, the altitude from R to line PQ must have height $h \leq 2\epsilon/b$ and so R must lie in a strip of width $4\epsilon/b$ and length at most $\sqrt{2}$. This occurs with probability at most $4\sqrt{2}\epsilon/b$. As $0 \leq b \leq \sqrt{2}$ the total probability is bounded by

$$\int_0^{\sqrt{2}} (2\pi b)(4\sqrt{2}\epsilon/b)db = 16\epsilon$$

Now let $P_1, \ldots, P_{2n}$ be selected uniformly and independently in S and let X denote the number of triangles $P_iP_jP_k$ with area less than $1/(100n^2)$. For each particular i, j, k the probability of this occuring is less than $.016n^{-2}$ and so

$$E[X] \leq \binom{2n}{3}(.016n^{-2}) < n$$

Thus there exists a specific set of $2n$ vertices with fewer than n triangles of area less than $1/(100n^2)$. Delete one vertex from the set from each such triangle. This leaves at least n vertices and now no triangle has area less than $1/(100n^2)$. $\square$

We note the following construction of Erdős showing $T(n) \geq 1/(2(n-1)^2)$ with n prime. On $[0, n-1] \times [0, n-1]$ consider the n points (x, x^2) where x^2 is reduced mod$n$. (More formally, (x, y) where $y \equiv x^2 \bmod n$ and $0 \leq y < n$.) If some three points of this set were collinear they would line on a line $y = mx + b$ and m would be a rational number with denominator less than n. But then in Z_n^2 the parabola $y = x^2$ would then intersect the line $y = mx + b$ in three points, so that the quadratic $x^2 - mx - b$ would have three distinct roots, an impossibility. Triangles between lattice points in the plane have as their areas either halfintegers or integers, hence the areas must be at least $1/2$. Contracting the plane by an $n-1$ factor in both coordinates gives the desired set. While this gem does better than Theorem 3.1 it does not lead to the improvements of Komlós, Pintz and Szemerédi.

4 Packing

Let C be a bounded measurable subset of R^d and let $B(x)$ denote the cube $[0,x]^d$ of side k. A *packing* of C into $B(x)$ is a family of mutually disjoint copies of C, all lying inside $B(x)$. Let $f(x)$ denote the largest size of such a family. The packing constant $\delta = \delta(C)$ is defined by

$$\delta(C) = \frac{1}{\mu(C)} \lim_{x \to \infty} f(x) x^{-d},$$

the maximal proportion of space that may be packed by copies of C. (This limit can be proven always to exist but even without that result the following result holds with lim replaced by lim inf.)

Theorem 4.1 Let C be bounded, convex, and centrally symmetric around the origin. Then

$$\delta(C) \geq 2^{-d-1}$$

Proof. Let P, Q be selected independently and uniformly from $B(k)$ and consider the event $(C+P) \cap (C+Q) \neq \emptyset$. For this to occur we must have, for some $c_1, c_2 \in C$

$$P - Q = c_1 - c_2 = 2\frac{c_1 - c_2}{2} \in 2C$$

by central symmetry and convexity. The event $P \in Q + 2C$ has probability at most $\mu(2C)x^{-d}$ for each given Q hence

$$\Pr[(C+P) \cap (C+Q) \neq \emptyset] \leq \mu(2C)x^{-d} = 2^d x^{-d} \mu(C)$$

Now let $P_1, \ldots, P_n$ be selected independently and uniformly from $B(k)$ and let X be the number of $i < j$ with $(C+P_i) \cap (C+P_j) \neq \emptyset$. From linearity of expectation

$$E[X] \leq \frac{n^2}{2} 2^d x^{-d} \mu(C)$$

Hence there exists a specific choice of n points with fewer than that many intersecting copies of C. For each P_i, P_j with $(C + P_i) \cap (C + P_j) \neq \emptyset$ remove either P_i or P_j from the set. This leaves at least $n - \frac{n^2}{2} 2^d x^{-d} \mu(C)$ nonintersecting copies of C. Set $n = x^d 2^{-d}/\mu(C)$ to maximize this quantity, so that there are at least $x^d 2^{-d-1}/\mu(C)$ nonintersecting copies of C. These don't all lie inside $B(x)$ but, letting w denote an upper bound on the absolute values of the coordinates of the points of C, they do all lie inside a cube of side $x + 2w$. Hence

$$f(x + 2w) \geq x^d 2^{-d-1}/\mu(C)$$

and so

$$\delta(C) \geq \lim_{x\to\infty} \frac{f(x + 2w)(x + 2w)^{-d}}{\mu(C)} \geq 2^{-d-1}$$

5 Recoloring

Suppose that a random coloring leaves a set of blemishes. Here we apply a random recoloring to the blemishes to remove them. If the recoloring is too weak then not all the blemishes are removed. If the recoloring is too strong then new blemishes are created. We use the notation of Chapter 1, §3 on Property B: $m(n) > m$ means that given any n-uniform hypergraph $H = (V, E)$ with m edges there exists a 2-coloring of V so that no edge is monochromatic. J. Beck [1978] proved $m(n) = \Omega(2^n n^{1/3})$. Here we use his method to give a slightly weaker result.
Theorem 5.1 If there exists p, $0 \leq p \leq 1/2$, with

$$2ke^{-pn} + k^2 pe^{pn} < 1$$

then $m(n) > 2^{n-1}k$.
Corollary 5.2 $m(n) = \Omega(2^{n-1} n^{1/3} (\ln n)^{-1/2})$
Proof. For $k < cn^{1/3}(\ln n)^{-1/2}$ with $c < \sqrt{3}$ take $p = (\ln n)/3n$.
Proof of Theorem 5.1 Fix $H = (V, E)$ with $m = 2^{n-1}k$ edges and p satisfying the condition. First color V randomly, each $v \in V$ is Red or Blue with independent probability .5. Call this the First Coloring. Now for each $v \in V$ that lies in a monochromatic $e \in E$ change the color of v with probability p. (Even if v lies in many monochromatic e the probability of its color changing is still precisely p. The color changes, of course, are done independently.) The new coloring is called the Recoloring.

For each $e \in E$ let A_e be the event that e is monochromatic in both the First Coloring and the Recoloring. Once e is monochromatic in the First

Coloring each $v \in e$ will "flip its coin" once again so that

$$\Pr[A_e] = 2^{1-n}(p^n + (1-p)^n) \le 2^{1-n}(2(1-p)^n)) \le 2^{1-n}(2e^{-pn})$$

so that

$$\Pr[\vee A_e] \le m2^{1-n}(2e^{-pn}) = 2ke^{-pn}$$

For each $e, f \in E$ with $e \cap f \neq \emptyset$ let A_{ef} be the event that e was Red in the First Coloring and f was Blue in the Recoloring, or the same with Red/Blue reversed. For every $W \subseteq f - e$ let A_{efW} be the event that $e \cup W$ was Red in the First Coloring, $f - (e \cap f) - W$ was Blue in the First Coloring, and f was Blue in the Recoloring - or the same with Red/Blue reversed. Then $A_{ef} = \vee A_{efW}$, the disjunction over all $W \subseteq f - e$.

Set $|e \cap f| = t$ and $|W| = w$. Then

$$\Pr[A_{efW}] \le 2 \cdot 2^{-2n+t} p^{t+w}$$

The factor 2 is the choice of Red or Blue. The factor 2^{-2n+t} is the probability of achieving the precise First Coloring of $e \cup f$. The probability p^{t+w} bounds the probability that every point of $(e \cap f) \cup W$ changes its color in the Recoloring. (p^{t+w} is an upper bound since also every point of W must lie in a set monochromatic under the First Coloring and also no other point of f can have its color changed. Attempts to improve Beck's result have centered at this point but have been, thus far, unsuccessful.) Summing over all W

$$\Pr[A_{ef}] = \sum_{W \subseteq f-e} \Pr[A_{efW}] \le \sum_{w=0}^{n-t} \binom{n-t}{w} (2^{1-2n+t}p^t)p^w = 2^{1-2n+t}p^t(1+p)^{n-t} \le 2^{1-2n}e^{pn}\left(\frac{2p}{1+p}\right)^t$$

As $p \le 1/2$, $2p/(1+p) < 1$ and the last expression is minimized when $t = 1$. Thus

$$\Pr[A_{ef}] \le 2^{1-2n}e^{pn}\frac{2p}{1+p} \le 2^{2-2n}e^{pn}p$$

There are less than m^2 intersecting pairs $e, f \in E$ so

$$\Pr[\vee A_{ef}] \le m^2 2^{2-2n} e^{pn} p = k^2 e^{pn} p$$

and hence

$$\Pr[\vee A_e \vee \vee A_{ef}] < 1$$

There is a First Coloring and Recoloring for which none of the A_e nor any of the A_{ef} hold. But we have covered all ways a set can be monochromatic under the Recoloring - hence there is a First Coloring and Recoloring for which there are no monochromatic sets in the Recoloring and so $m(n) > m$.
□