Fundamental Algorithms, Assignment 11

Solutions

1. Consider `Dumb Prim` for MST. The high level idea is the same but to find the minimal weight of an edge $\{i, j\}$, $i \in S$, $j \notin S$, one looks at all the weights and finds the minimum in the usual way. Assume that all pairs $\{i, j\}$ have a weight. Let n be the number of vertices.

   (a) When $|S| = i$ what is the time to add a vertex to S as a function of n and i.
   `Solution:`$O(i(n-i))$ as you need the minimum of that many terms. (Actually, to get this you can't be "too dumb." One way is to keep S and $\overline{S}$ in linked lists. When x moves from $\overline{S}$ to S it takes $O(n)$ to remove it from $\overline{S}$ and $O(1)$ to add it to S. Now initialize $MIN = \infty$ and do a double loop on S and $\overline{S}$ to find that i, j, $i \in S$, $j \in \overline{S}$ with minimal weight.)

   (b) What is the total time for `Dumb Prim` as a sum over i.
   `Solution:`$O(\sum_{i=1}^{n-1} i(n-i))$ as you start with $|S| = 1$ and end with $|S| = n-1$.

   (c) Evaluate the above sum as $\Theta(g(n))$ for some nice function $g(n)$. (Caution: The time is *not* an increasing function of i. For example, when $i = n-1$ the time is quite quick.)
   `Solution:`$O(n^3)$. The biggest term is the middle $i = n/2$ with $i(n-i) = n^2/4$ and there are $n-1$ terms so the sum is at most $(n-1)n^2/4 \sim n^3/4$. The $i = n/4$ term gives $i(n-i) = 3n^2/16$ and all terms from $i = n/4$ to $i = 3n/4$ are at least that big so the sume is at least $(n/2)(3n^2/16) \sim (3/32)n^3$. We've sandwiched the sum so it is $\Theta(n^3)$.

   (d) Compare the time for `Dumb Prim` with `Prim` as discussed in class
   `Solution:Prim` takes $O(E \ln V)$. Here $V = n$ and $E = \binom{n}{2} \sim n^2/2$ so Prim takes $O(n^2 \ln n)$, definitely faster than `Dumb Prim`.

2. Consider Prim's Algorithm for MST on the complete graph with vertex set $\{1, \ldots, n\}$. Assume that edge $\{i, j\}$ has weight $(j-i)^2$. Let the root vertex $r = 1$. Show the pattern as Prim's Algorithm is applied.
   `Solution:`The set S, initially $\{1\}$, will grow to $\{1, 2\}$,…,$\{1, 2, \ldots, i\}$, …, $\{1, \ldots, n\}$. When $S = \{1, \ldots, i\}$ the closest point to S will be $i+1$ with $\pi[i+1] = i$ and $key[i+1] = 1$. In particular, Let $n = 100$ and consider the situation when the tree created has 73 elements and π and key have been updated.

(a) What are these 73 elements.
`Solution:`$1, \ldots, 73$

(b) What are $\pi[84]$ and $key[84]$.
`Solution:`$\pi[84] = 73$ (all other of $1, \ldots, 72$ are further) and $key[84] = (84-73)^2 = 121$.

3. Find $d = \gcd(89, 55)$ and x, y with $89x + 55y = 1$. [Remark: This is part of a pattern with two consecutive numbers from the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$]
`Solution:`

$$\texttt{EUCLID}(89, 55) = \texttt{EUCLID}(55, 34) = \texttt{EUCLID}(34, 21) =$$
$$= \texttt{EUCLID}(21, 13) = \texttt{EUCLID}(13, 8) = \texttt{EUCLID}(8, 5) =$$
$$= \texttt{EUCLID}(5, 3) = \texttt{EUCLID}(3, 2) = \texttt{EUCLID}(2, 1) =$$
$$= \texttt{EUCLID}(1, 0) = 1$$

with all quotients 1 except the last. For $\texttt{EXTENDED} - \texttt{EUCLID}$ we get a chart like Figure 31.1:

a	b	$\lfloor a/b \rfloor$	d	x	y
89	55	1	1	-21	34
55	34	1	1	13	-21
34	21	1	1	-8	13
21	13	1	1	5	-8
13	8	1	1	-3	5
8	5	1	1	2	-3
5	3	1	1	-1	2
3	2	1	1	1	-1
2	1	2	1	0	1
1	0	-	1	1	0

so $x = -21$ and $y = 34$. (Note that the x's and y's form a Fibonacci like pattern as well!)

4. Find $\frac{211}{507}$ in Z_{1000}.
`Solution:`Here we first find $\texttt{EUCLID}(1000, 507)$:

$$\texttt{EUCLID}(1000, 507) = \texttt{EUCLID}(507, 493) = \texttt{EUCLID}(493, 14) =$$
$$= \texttt{EUCLID}(14, 3) = \texttt{EUCLID}(3, 2) = \texttt{EUCLID}(2, 1) =$$

$$= \texttt{EUCLID}(1, 0) = 1$$

For $\texttt{EXTENDED} - \texttt{EUCLID}$ we get a chart like Figure 31.1:

a	b	$\lfloor a/b \rfloor$	d	x	y
1000	507	1	1	181	-357
507	493	1	1	-176	181
493	14	1	35	5	-176
14	3	1	4	-1	5
3	2	1	1	1	-1
2	1	1	2	0	1
1	0	-	1	1	0

so that $1000(181) - 357(507) = 1$ so in Z_{1000} we have $(-357)(507) = 1$ so $\frac{1}{507} = -357 = 643$. Finally $\frac{211}{507} = 211 \cdot 643 = 135673 = 673$. So the answer is 673. To check: $673 \cdot 507 = 341211 = 211$.

5. Solve the system
   $x \equiv 34 \bmod 101$
   $x \equiv 59 \bmod 103$.
   `Solution:`We write $x = 103y + 59$ (we could start with either and this one is a bit easier) so that in Z_{101} we want $103y + 59 = 34$ or $2y = -25 = 76$ and $y = 38$. (Usually division is complicated but here it worked out like normal division.) Then $x = 103(38) + 59 = 3973$. The general answer is given as $x \equiv 3973 \bmod 10403$ as $10403 = 103 \cdot 101$.

6. Using the Island-Hopping Method to find 2^{1000} modulo 1001 using a Calculator but NOT using multiple precision arithmetic.
   `Solution:`

$$2^1 = 2$$

$$2^2 = 2 \cdot 2 = 4$$

$$2^4 = 4 \cdot 4 = 16$$

$$2^8 = 16 \cdot 16 = 256$$

$$2^{16} = 256 \cdot 256 = 65536 = 471$$

$$2^{32} = 471 \cdot 471 = 221841 = 620$$

$$2^{64} = 620 \cdot 620 = 384400 = 16$$

$$2^{128} = 16 \cdot 16 = 256$$

$$2^{256} = 256 \cdot 256 = 65536 = 471$$

$$2^{512} = 471 \cdot 471 = 221841 = 620$$

As $1000 = 512 + 256 + 128 + 64 + 32 + 8$ we have

$$2^{1000} = 620 \cdot 471 \cdot 256 \cdot 16 \cdot 620 \cdot 256$$

We calculate in stages: $620 \cdot 471 = 292020 = 729$, $729 \cdot 256 = 186624 = 438$, $438 \cdot 16 = 7008 = 1$, $1 \cdot 620 = 620 = 620$, $620 \cdot 256 = 158720 = 562$, So the answer is 562. This shows that 1001 is *definitely* not a prime. (Of course, for numbers this small there are easier ways!)

7. Suppose that during Kruskal's Algorithm (for MST) and some point we have $SIZE[v] = 37$. What is the interpretation of that in the case when $\pi[v] = v$?
   `Solution:`At that moment v is in a component of size 37 and it is the root of the associated tree.
   What is the interpretation of that in the case when $\pi[v] = u \neq v$?
   `Solution:`v had had size 37 at the moment when $\pi[v]$ was changed, and the component with v was joined to the (larger) component with u.
   How many different values can $\pi[w]$ have during the course of Kruskal's algorithm?
   `Solution:`Two. Initially $\pi[w] = w$ but once it changes to $\pi[w] = v$ it doesn't change any more. Precisely one vertex does not ever change, it becomes the root of the final rooted tree.
   How many different values (as a function of V, the number of vertices) can $SIZE[w]$ have during the course of Kruskal's algorithm?
   `Solution:`V. It is possible that w is joined to isolated vertices $V - 1$ times and so $SIZE[w]$ goes from 1 to V by ones.