Fundamental Algorithms
FINAL EXAM

There will be little rubs and disappointments everywhere, and we are all apt to expect too much; but then, if one scheme of happiness fails, human nature turns to another; if the first calculation is wrong, we make a second better.
– from Mansfield Park by Jane Austen

1. (20) Describe the algorithm `TOPDOLLAR(`$PRICE$`,`N`)`. The input is an array of nonnegative real numbers $PRICE[1 \cdots N]$ where $PRICE[I]$ represents the price for a rod of length I. Your output should be $VALUE$, which should be the optimal total price among all cuttings of the rod. You may, and should, use an auxilliary array. You may write in pseudocode or actual code but in either case you **must** give clear comments describing what you are doing and *why* it works. Analyze (just the answer will not suffice!), in Θ-land, the time for your algorithm.

2. (5) For n large which is faster: $\Theta(\lg^3 n)$ or $\Theta(\sqrt{n})$ algorithm?

3. (15) Given $A[1 \cdots N]$ with $0 \leq A[I] < N^N$ for all I.

   (a) How long will `COUNTING-SORT` take?

   (b) How long will `RADIX-SORT` take using base N?

   (c) Give an algorithm (no proof required!) that does better than both of the above, and does so in worst case.

4. (5) In a max Heap what is the property of the value at the root? (That is, how does it compare to the values of the other nodes.)

5. (20) Let G be a connected graph on vertex set V. Let a nonnegative weight $w(e)$ be assigned to every edge e. Let $S \subset V$ with $S \neq \emptyset$ and $S \neq V$.

   (a) Prove (yes, prove!) that the Minimal Spanning Tree necessarily contains that edge $e = \{x, y\}$ with $x \in S$ and $y \notin S$ which has minimal weight. A good picture won't hurt. Assume no two edges have the same weight. (Warning: Saying that an algorithm tells you to pick this edge e is **not** a proof!)

   (b) In what algorithm is the above result used.

6. (10) Consider the recursion $T(2n) = 3T(n) + n + 1$ with initial value $T(1) = 5$.

   (a) What is $T(4)$?

   (b) What it $T(n)$ in Θ-land.

7. (20) If multiplication mod p takes 1 second then show, using Island-Hopping, how to calculate x^{1023} quickly. How many seconds did it take? (Hint: What is special about 1023?)

8. (20) Find the Huffman code on letters a, b, c, d, e, f with frequencies $.13, .16, .09, .45, .05, .12$ respectively. Give pictures and words clearly indicating the intermediate steps in finding the code.

9. (5) State the Chinese Remainder Theorem.

10. (15) We call a positive integer n `SQUAREFREE` if there is no integer $d > 1$ with d^2 dividing n. We call n `NOTSQUAREFREE` if n is not `SQUAREFREE`.

    (a) Argue `NOTSQUAREFREE` is in `NP`.

    (b) (This is harder!) Argue `SQUAREFREE` is in `NP`.

11. (5) What was the breakthrough of Agrawal, Kayal and Saxena?

12. (20) In Depth-First Search each vertex v gets a discovery time $d[v]$ and a finishing time $f[v]$. Let G be a graph and v, w two distinct vertices for which $w \in Adj[v]$.

    (a) Assume $d[v] < d[w]$. Argue that $f[w] < f[v]$.

    (b) Now further assume that G is a DAG (a Directed Acyclic Graph.) Assume $d[w] < d[v]$. Argue that $f[w] < f[v]$.

13. (5) What is the difference, if any, between a $\Theta(\lg n)$ algorithm and a $\Theta(\log_{10} n)$ algorithm? (*Short* reason please.)

    In the end the great truth will have been learned, that the quest is greater than what is sought, the effort finer than the prize, or rather that the effort is the prize, the victory cheap and hollow were it not for the rigor of the game.
    – Justice Benjamin Cardozo