Note on Rod Cutting

This is covered in §15.1. Here is Prof. Spencer's view of it.

We can sell a rod of length I for $P[I]$. These values are given to us. (Everything here is integral.)

Now we have a rod of length N. How can we best cut the rod into pieces so as to maximize our revenue?

There are useful heuristics for this but here we give a method, a form of *dynamic programming* that gives the exact answer and does it in time $O(N^2)$.

We create an array $R[S]$ which will be the maximal total revenue we can get starting with a rod of length S. While our goal is to find $R[N]$ our method is to "work up" to this goal by finding $R[1], R[2], \ldots$ until we reach $R[N]$. We initialize with $R[0] = 0$. (If you like, set $R[1] = P[1]$ as well.) So our program will start:

$R[0] = 0$

FOR $S = 1$ to N (* Now want to find $R[S]$ *)

We want (in the guts of the FOR loop) to find $R[S]$ where we *already know* $R[0], R[1], \ldots, R[S-1]$. The key is to think about the first cut of the rod. We don't know where we should make it, it will be at some I where $1 \leq I \leq S$. ($I = S$ would be selling the entire rod as a single piece.) *Suppose* we did cut it at I so we would receive revenue $P[I]$ for the first piece. The remaining rod now has length $S - I$. We would now (and this is a feature of dynamic programming) want to cut up that piece so as to get the maximal revenue but we *already know* that we will get $R[S-I]$ from that piece. So then our total revenue would be $P[I] + R[S-I]$. (Note that if we sell the rod of length S as a single piece we get $P[S] + R[0] = P[S]$ so this is included.)

Which I should we choose for the first cut? Try them all! Pick that I which gives the maximal value of $P[I] + R[S-1]$. Finding a max takes time $O(S)$, with a single loop:

$MAX = 0$

FOR $I = 1$ to S

    IF $P[I] + R[S-I] \geq MAX$ THEN $MAX \leftarrow P[I] + R[S-I]$

END FOR

This MAX will be our value for $R[S]$. Here is the whole program. It is a double loop and the time is $O(N^2)$.

$R[0] = 0$
FOR $S = 1$ to N
  $MAX = 0$
  FOR $I = 1$ to S
    IF $P[I] + R[S - I] \geq MAX$ THEN $MAX \leftarrow P[I] + R[S - I]$
  END FOR
  $R[S] \leftarrow MAX$
END FOR
RETURN $R[N]$

What if you want to actually find the optimal cut? When we are calculating $R[S]$ we find that I which does maximaize $P[I] + R[S - I]$. We do this by having another array $FIRSTCUT[S]$. We modify the calculation of MAX by:

$MAX = 0$
FOR $I = 1$ to S
  IF $P[I] + R[S - I] \geq MAX$ THEN
    $MAX \leftarrow P[I] + R[S - I]$
    $FIRSTCUT[S] = I$
END FOR

In this approach $FIRSTCUT[S]$ keeps changing but its last value (the one that sticks) is that I with $P[I] + R[S - I] = MAX$.

Now to print out the cuts for N we (REM denotes the remaining part of the rod):

$REM = N$
WHILE $REM > 0$
  PRINT $FIRSTCUT[REM]$
  $REM \leftarrow REM - FIRSTCUT[REM]$
END WHILE