The QuickSort/BST recursion

We here consider the recursion

$$T(n) = n - 1 + \frac{2}{n}\sum_{i=1}^{n-1} T(i) \tag{1}$$

with initial conditions $T(0) = T(1) = 0$. The recursion comes up in QuickSort and in random BST, but this is a mathpage.

We first multiply out

$$nT(n) = n(n-1) + 2\sum_{i=1}^{n-1} T(i) \tag{2}$$

Now we replace n with $n+1$

$$(n+1)T(n+1) = (n+1)n + 2\sum_{i=1}^{n} T(i) \tag{3}$$

Subtracting (2) from (3) gives

$$(n+1)T(n+1) - nT(n) = 2n + 2T(n) \tag{4}$$

or

$$(n+1)T(n+1) = (n+2)T(n) + 2n \tag{5}$$

Dividing (5) by $n+1)(n+2)$:

$$\frac{T(n+1)}{n+2} = \frac{T(n)}{n+1} + \frac{2n}{(n+1)(n+2)} \tag{6}$$

Now we define the auxilliary function

$$S(n) = \frac{T(n)}{n+1} \tag{7}$$

so that (6) becomes

$$S(n+1) = S(n) + \frac{2n}{(n+1)(n+2)} \tag{8}$$

with initial conditions $S(0) = S(1) = 0$. Thus $S(n)$ is the sum:

$$S(n) = \sum_{i=1}^{n-1} \frac{2i}{(i+1)(i+2)} \tag{9}$$

As the addends are $\sim 2i^{-1}$ the sum is $\sim 2\ln n$. So

$$S(n) \sim 2\ln n \tag{10}$$

and

$$T(n) \sim 2n\ln n \tag{11}$$

We can be more precise. Using partial fractions we write

$$\frac{2i}{(i+1)(i+2)} = \frac{4}{i+2} - \frac{2}{i+1} \tag{12}$$

We express our answer in terms of the *harmonic sum*

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \tag{13}$$

so that

$$\sum_{i=1}^{n-1} \frac{1}{i+2} = H_n + \frac{1}{n+1} - 1 - \frac{1}{2} \tag{14}$$

and

$$\sum_{i=1}^{n-1} \frac{1}{i+1} = H_n - 1 \tag{15}$$

so that (9) becomes

$$S(n_{=} 2H_n - \frac{4n}{n+1} \tag{16}$$

and the final answer is quite clean:

$$T(n) = 2(n+1)H_n - 4n \tag{17}$$

Whew! Must recursions don't end up with such a nice answer!