An isoperimetric inequality for self-intersecting polygons

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Abstract

Let P be a non-simple polygon in R^2 with segments that are directed by a traversal along its vertices. Let $\mathcal{Z}$ be a multiset of oriented simple polygons (cycles) built from an in-place decomposition and regrouping of the directed located segments in P.

For any $\mathbf{x}$ in $ConvHll(P)$, let $w(\mathbf{x})$ be the maximum of 1 and the number of cycles in $\mathcal{Z}$ that contain $\mathbf{x}$ in their convex hulls. For any cycle C in $\mathcal{Z}$, let $W_+(C)$ and $W_-(C)$ be the number of cycles in $\mathcal{Z}$ that contain C their convex hulls and have, respectively the same/opposite orientation as C. Let $\kappa = 2\sqrt{2} - 2$, and $W(C) = W_+(C) + \kappa \cdot W_-(C)$.

Let a be the area of the largest polygon that can be constructed from translations of the segments in P. Then

$$\int_{\mathbf{x} \in ConvHll(P)} w^2(\mathbf{x}) d\mathbf{x} + \sum_{C \in \mathcal{Z}} W(C)(Area(ConvHll(C)) - Area(C)) \leq a.$$

With the exception of κ, no subexpression can be increased by a constant factor, and κ cannot exceed $\frac{\sqrt{13}-1}{3} \approx .869$. Previous bounds used the multiplier $w(\mathbf{x})$ rather than w^2 and set $W = 0$, or set $W = 0$ and replaced a with $\frac{p^2}{4\pi}$, where $p = Arclength(P)$. This latter formulation is elementary, and can be strengthened with, effectively, κ set to 1.

1 Introduction

Suppose a string of length l is wrapped with k full rotations into a circular shape. Evidently, the circumference of the circle is $\frac{l}{k}$, and its area is $\frac{l^2}{4\pi k^2}$. Thus, unwinding the string to form a simple circle with circumference l increases the enclosed area by a factor of k^2. Of course, this rescaling relationship extends to any figure, and we should expect this property to generalize. Suppose P is a polygon that is not simple. Let $R^2 \setminus P$ have k bounded connected components $P_1, P_2, \ldots, P_k$. Intuitively, the unsigned winding number w_i of a component P_i is the minimum, among all continuous paths from P_i to the unbounded component of $R^2 \setminus P$, of the number of crossings that the path has with P. This winding number is used to show that the edges of a polygon P can be rearranged to form a simple polygon whose area is at least

$$Area(ConvHll(P)) + \sum_i (w_i^2 - 1) Area(P_i).$$

Additional geometric characteristics can strengthen this bound. For example, suppose the string forms a semicircle that comprises a closed loop with no area. Taking the convex hull of this point-set produces a region with just half the area of a full circle. Similarly, the flip of a concavity in the boundary of a simple polygon increases the area by twice the gain that is attained by replacing the errant portion of the boundary with a straight support segment.

However, when the polygon is no longer simple, the flips can become more subtle, and this property can sometimes fail to hold. Still, a rearrangement of the boundary segments of a polygonal region R ought to yield an area of at least $2(Area(ConvHll(R)) - Area(R))$ if, say, R has a consistent orientation. And if R is just part of a more complicated, overlapping figure, then the factor of 2 should be increased to reflect additional area gains.

1.1 Basic Definitions

Technically, a polygon P is a finite number of segments connected end-to-end to form a closed path. If P is a simple polygon, let $|P|$ denote the area of the region bounded by P. If P is a region, let $|P|$ also denote its area. If ρ is a scalar,

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let ρP denote a rescaling (dilation) of P by the magnification factor ρ, so that $|\rho P| = \rho^2|P|$.

The difficulty with standard winding numbers is that they are signed, and can be zero for points of a region that are fully enclosed by a fixed boundary. Consequently, we must define a better winding metric. The metric will be relative to various decompositions of P, including one that is canonical.

Let P be a finite polygonal curve in R^2. Let each segment in P be directed according to a fixed traversal of P. We say that the point-set P_0 is a simple subcycle of P if P_0 is a simple cycle, P_0 is contained in P, and P_0 becomes an oriented cycle if each segment of P_0 is directed in the same direction as its containing segment in P.

The use of the word point-set in this definition is intentional. A segment in P_0 might comprise only a portion of a segment that belongs to P. Similarly, the sequencing of edges in P_0 might not be the same as in P. However, each subsegment will be oriented in the same direction as the full segment.

An oriented simple subcycle is negative if it winds about its interior in a clockwise direction, and is positive otherwise.

We extend this definition to *semisimple* cycles, which include two kinds of degenerate limit cases.

The first degeneracy extends a semisimple cycle to include an open chain or even a tree of simple cycles, which all have the same orientation. Each pair of cycles must either be disjoint or intersect at a single point. This intersection structure implicitly defines a graph where each simple cycle is represented by a vertex, and an edge represents a pair of cycles that intersect. The definition of semisimple cycles requires the graph to be acyclic and connected. Indeed, if the graph were not acyclic, then the edge collection would contain both a positive and a negative subcycle.

The second degeneracy accounts for cycles with an empty interior. The orientation of such a cycle S can be interpreted as being either positive or negative. In this case, we do not care what orientation it has, but do require any positive-length segment $e \subset S$ to be covered just twice by elements in S. These degenerate cycles can be subcycles within a semisimple cycle.

A decomposition of P is a collection of oriented located semisimple cycles whose union defines the same multiset of directed segments as the those in P.

The canonical decomposition of P is based on maximal semisimple subcycles. An oriented simple subcycle P_0 is maximal if it is not contained in the boundary plus interior of some other comparably oriented simple subcycle of P. The decomposition is as follows: Repeatedly identify some maximal subcycle in P and remove it. For completeness, it should be observed that if several cycles are congruent and overlap perfectly, it suffices to select one copy per iteration.

It is not difficult to show that, apart from the semisimple subcycles with zero area, this decomposition of P is independent of the ordering of the extraction steps for the maximal positive and negative subcycles. The decomposition can be made unique by treating subcycles with an empty interior as third cycle type that cannot be combined with positively or negatively oriented semisimple cycles.

Let $\mathcal{Z}$ be a decomposition of P into semisimple subcycles.

Let, unless otherwise stated, κ be the constant $\sqrt{2} - 2$.

For any $\mathbf{x}$ in $ConvHll(P)$, let $w(\mathbf{x})$ be the maximum of 1 and the number of cycles in $\mathcal{Z}$ that contain $\mathbf{x}$ in their convex hulls.

For any simple cycle C, let $W_+(C)$ and $W_-(C)$ be the number of cycles in $\mathcal{Z}$ that contain C their convex hulls and have, respectively the same/opposite orientation as C. Let $W(C) = W_+(C) + \kappa \cdot W_-(C)$.

Let A_P be the area of the largest polygon with edges that can be formed from segments having the same multiset of lengths as P.

Theorem 1. With the parameters as defined:

$$\int_{\mathbf{x}\in ConvHll(P)} w(\mathbf{x})^2 d\mathbf{x} + \sum_{C\in\mathcal{Z}} W(C)(|ConvHll(C)| - |C|) \leq A_P.$$

The proof requires some development that begins in Section 2, where a slightly stronger bound is formulated in terms of the area of the largest polygon that can be constructed from translations (without rotation) of the segments in P.

1.2 Prior results

The standard Isoperimetric Theorem for polygons with a fixed set of edge lengths is as follows (cf. [4]):

Theorem A. Let P be a simple polygon. Then among all simple polygons with edge lengths that are the same as those of P, those that can be inscribed in a circle have the greatest area.

This result dates back 22 centuries to the ancient Greeks. On the other hand, the historical record seems to show that there were no correct proofs until the latter half of the 19^{th} Century [4], and generalizations of this bound are still a matter of interest. For example, suppose that P is a polygon that is not simple. Let the bounded components of $R^2 \setminus P$ be $\{P_i\}$. In 1947, Radó proved a bound of the form [7]:

$$\sum_i |w_i| Area(P_i) \leq A_\circ,$$

where w_i is the winding number of points in P_i with respect to P and $A_\circ$ is the area of a circle with perimeter equal to that of P. Osserman [5] points out that the bound is also a special case of an isoperimetric inequality established by Federer and Fleming [3, 1, Cor. 6.5 and Remark 6.6]. In 1968, Pach [6] showed that $Area(ConvHll(P)) \leq A_P$, where A_P is the area of the largest polygon with sides congruent to those of P, and is as characterized in Theorem A.

In 1986, Böröczky, Bárány, Makai, and Pach proved, among other things, a bound [2] of the form

$$\int_{\mathbf{x} \in ConvHll(P)} w(\mathbf{x}) d\mathbf{x} \leq |Sor(P)|,$$

where $w(\mathbf{x})$ is defined in Section 1.1, and $|Sor(P)|$, as formalized in Section 2, is the area of the largest polygon that can be constructed from translations (without rotation) of the segments in P. In 1971, Banchoff and Pohl [1] improved the Radó formulation to use the multipliers w_i^2:

$$\sum_i w_i^2 Area(P_i) \leq A_\circ,$$

In Theorem 2, we combine the strongest characteristics of both the Banchoff and Pohl bound and that of Böröczky, Bárány, Makai, and Pach. From a technical perspective, the use of $|Sor(P)|$ as opposed to $A_\circ$, and the inclusion of the convex hull pose impediments to stronger bounds. For example, the orientation of cycles is irrelevant when the target area bound is defined in terms of $A_\circ$, because the direction of a cycle can be reversed, and the new figure with its new boundary will have the same arclength, and will define the same components in the plane. Such a cut-and-reverse-connections approach can be combined with the Brunn-Minkowski inequality to prove a bound comparable to that of Banchoff and Pohl as follows.

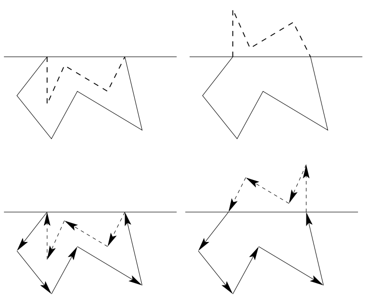
Let P_1 be a polygon that is not simple. An equivalent polygon with positive orientation can be defined in terms of the following decomposition. Given P_i, let U_i be the unbounded component of of $R^2 \setminus P_i$; set $Q_i = \partial U_i$ with the orientation taken to be counterclockwise. Let P_{i+1} comprise P_i minus those segments pieces that (apart from orientation) were combined in-place to produce Q_i. The equivalent polygon is $Q = \cup Q_i$.

Then convexifying flips (or the equivalent as formalized in the beginning of Section 2) plus the Brunn-Minkowski inequality can be used to show that $2|ConvHll(Q) \cap U_1| + \sum_{i \geq 1} i^2 |Q_i \cap U_{i+1}| \leq \frac{p^2}{4\pi}$, where P_1 has an arclength equal to p. Inasmuch as the derivation is straightforward, we omit the details.

However, this inequality and the Brunn-Minkowski inequality by itself seem inadequate to establish Theorems 1 and 2, and the use of cycle reversals seems inappropriate for cycles with crossing edges in this case. These theorems require that the edge set must be preserved without breaks that might otherwise increase the target area bounds of A_P and $|Sor(P)|$.

2 Preliminaries

Let X be an oriented simple polygon. If the figure is not convex, we can find a global support line $\ell = \overleftrightarrow{v_i v_j}$, which intersects X at two vertices and includes all of X in one of its (closed) halfplanes. The vertices v_i and v_j split X into two subpaths.



Suppose that $X \cap \overline{v_i v_j} = \{v_i, v_j\}$, so that $\overline{v_i v_i}$ seals off some pocket of X as shown. Steiner symmetrization flips (reflects) one of the subpaths about ℓ to increase the area of the resulting figure while preserving the edge lengths. As a practical matter, it is simpler to reverse the sequencing of these edges, which effectively rotates the boundary portion about the midpoint of $\overline{v_i v_j}$ (but does not reverse their direction). While iterations of either operation lead, eventually, to a convex region with increased area, the virtual rotation operation is easier to analyze. It simply rearranges the segment ordering without changing their directions. Since this procedure can produce no more than $(n-1)!$ such arrangements, one of these polygons must have a maximal area. This polygon must be convex, since otherwise this procedure could further increase the area.

Definition. Let ℓ be a ray or directed line. The *angular direction* of ℓ is the measure of the angle formed by ℓ and a horizontal ray h that originates at some point on ℓ and runs to the right.

For specificity, we take the direction of an angle to be counterclockwise as measured from h to ℓ. Thus, angular directions are unique (mod 2π).

Definition. Let X be an oriented polygon. Let $Sor(X)$ be the figure that is formed by translating the edges of X (without rotation) to produce a connected chain with the edges ordered according to a sorting by their angular direction. If X is simple, we will require that $Sor(X)$ have the same orientation as X.

Since X defines a closed curve (or if X comprises a collection of closed curves), the vectors that represent the edges sum to zero, and $Sor(X)$ will, therefore, be a closed polygon.

It should be clear that the resulting figure is convex. Evidently, $Sor(X)$ is the only translation-based edge rearrangement that results in a convex polygon (up to a reversal in orientation that is given by reverse sequencing the segments in $Sor(X)$).

The Sor transformation can also be applied to any set of directed edges. Formally, if Y is a path, $Sor(Y)$ can be defined by adjoining, to Y, an additional segment that produces a polygon, applying the Sor, and then deleting the new edge from the convex figure.

We will actually show that for any decomposition of P:

Theorem 2. With the definitions as stated for Theorem 1,

$$\int_{\mathbf{x}\in ConvHll(P)} w(\mathbf{x})^2 d\mathbf{x} + \sum_{C\in\mathcal{Z}} W(C)(|ConvHll(C)| - |C|) \le |Sor(P)|.$$

This inequality is a little stronger than our target bound, but is technically preferable to prove. With this formulation, which prohibits rotations, we are free to break and rearrange individual segments and use each piece any way we please in intermediate constructions. Once the final edge collection is sorted by direction, all of the pieces will come back together to reform a translation of original segment.

Definition. Let X and Y be polygons. Let $X \cup Y$ represent both a collection of edges that comprise the union of segment sets in X and Y, and (when the context is appropriate) the union of the physical point-sets denoted by the polygons X and Y.

Let X and Y be polygons. Define $X \oplus Y$ to be $Sor(X \cup Y)$.

For convex polygons X and Y, $X \oplus Y$ is essentially equivalent to the traditional Minkowski sum in the sense that $ConvHll(X \oplus Y)$ is the Minkowski sum of $ConvHll(X)$ and $ConvHll(Y)$. However, there is one important difference. If X and Y are directed with opposite orientations, then the definition is ambiguous in that $Sor(X \cup Y)$ represents the corresponding Minkowski sum where one of the two polygons has its edges reverse sequenced. This reversal gives a polygon with the same collection of directed edges, but an opposite orientation. Then the two polygons can be combined via the usual definitions for the Minkowski sum. Of course, we never use this construction explicitly, and need not resolve the ambiguity in terms of what orientation the resulting polygon should have.

Let P be a convex polygon, and let ℓ_L and ℓ_R be, respectively, vertical support lines on the left and right sides of P. These lines split P into an upper boundary and a lower boundary. If a segment of P is collinear with a support line, we can arbitrarily split the edge into two segments, and declare that the upper piece belongs to the upper boundary, and likewise for the lower piece. Let $\hat{P}$ be the upper boundary curve of P and $\check{P}$ the lower.

Let the edges of $\hat{P}$ be $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{\mathrm{j}}$, and let the edges of $\check{P}$ be defined analogously. A basic fact, which is easily verified, is that

$$|ConvHll(\hat{P})| = \frac{1}{2}\sum_{h<i\leq \hat{\jmath}} |\hat{e}_h \times \hat{e}_i|,$$

where the operator $\times$ denotes the vector cross product.

If Q and R represent collections of directed edges, let $Q * R = \frac{1}{2}\sum_{e\in Q,\ f\in R} |e \times f|$.

Let Q and R be convex polygons with upper and lower boundaries $\hat{Q}$, $\check{Q}$, $\hat{R}$ and $\check{R}$. Evidently,

$$|Q \oplus R| = |Q| + |R| + \hat{Q} * \hat{R} + \check{Q} * \check{R}.$$

Similarly,

$$|Q| = \frac{1}{2}(\hat{Q} * \hat{Q} + \check{Q} * \check{Q}).$$

The major focus of the proof is to attribute area coverage to the appropriate pairs of edges that belong to different cycles. The chief difficulties will be to account for the area increases due to the convex hull, and to devise a suitable organization for the collection of edges that form the region. Once these issues are resolved, the rest of the proof will be simplified by various inequalities of the Brunn-Minkowski type for two dimensions. The basic Brunn-Minkowski inequality is as follows:

Theorem B [Brunn,Minkowski] Let Q and R be convex polygons. Then

$$|Q \oplus R| \geq |Q| + 2\sqrt{|Q|\cdot|R|} + |R|.$$

However, Theorem 2 seems to require a slightly stronger formulation that is stated below in Lemma 1. Theorem B will be established as a corollary of Lemma 1.

Definition. Let Q be a connected region. Define the *horizontal diameter* of Q to be the width of the smallest infinite vertical strip that contains Q.

Lemma 1. Let Q and R be oriented convex polygons with the equal horizontal diameters. Then

$$|Q \oplus R| \geq 2|Q| + 2|R|.$$

Proof: Suppose the proposition is false for polygons Q and R. The basic idea is to use simple reshaping procedures to change Q and R in a way that increases $\Delta \equiv 2|Q| + 2|R| - |Q \oplus R|$ and yields regions where the corresponding Δ is easily seen to be zero.

Let $U = Q \oplus R$. Let the edges of Q, R, and U be $q_1, q_2, \ldots, q_h$, $r_1 \ldots r_i$, $u_1, \ldots, u_{h+i}$. In particular, the edges are connected in consecutive order to form their respective polygons, and a single side of U might be represented by two segments u_t and u_{t+1}, if the corresponding edges for Q and R are parallel.

Suppose q_t does not intersect a vertical support line of Q, and suppose q_t is not parallel to any edge in R. Let u_s correspond to q_t. Remove u_s from the figure U while keeping all other segments in place. Extend u_{s-1} and u_{s+1} to intersect. Replace q_s by the two extensions to the original segments u_{s-1} and u_{s+1}. Note that if u_{s-1} and u_{s+1} fail to intersect on the remote side of u_s, then u_s must touch a vertical support line of U, whence q_t must also do the same with respect to Q. It is easy to see that this construction increases $|U|$ and $|Q|$ by the same amount, and hence Δ must increase.

This procedure can also be applied to an edge that touches a vertical support line provided it is not vertical. Suppose, for example, that u_s touches a vertical support line on the left, and is connected to u_{s-1} on the right. In this case, the vertical support line can be treated as u_{s+1}, with u_{s-1} extended to meet it.

These transformations can be applied to produce new regions Q and R where every edge that is not vertical in one polygon has a corresponding parallel representative in the other.

Now let q_s be an edge in Q that is not vertical, and let $r_t \in R$ be parallel to q_s. Suppose q_s is longer than r_t. Let the lines ℓ_Q and ℓ_R be parallel to q_s and r_t, have distance ϵ, for sufficiently small ϵ, from their respective segments, and be located so that ℓ_R intersects R and ℓ_Q is exterior to Q. Extend (or shorten) the sides adjacent to q_s and r_t so that they terminate at ℓ_Q and ℓ_R. Let q_s and q_t be replaced by the corresponding segments of ℓ_Q and ℓ_R as delineated by the modified sides of R and Q. These modifications change Q and R in a way where $Q \oplus R$ is unchanged, but $|Q| + |R|$

has increased, and the two new sides are closer in length. (The construction will fail only if q_t is connected to a vertical segment and r_s has no vertical connection on the corresponding side in R.)

It follows that wherever this scheme can be applied, corresponding parallel segments can be made to have equal lengths. By inserting vertical segments of length 1, say, on both sides of Q and R, and corresponding segments of length 2 for U, the non-issue about non-vertical sides that attach to vertical edges can be resolved. The insertion scheme preserves Δ. Consequently, Q and R can be transformed so that each pair of parallel segments (that are not vertical) have equal length. Corresponding vertical segments for Q and R need not be equal, but the difference in the heights of the two corresponding pairs of vertical segments must be the same, since all other pairs of corresponding sides are congruent and parallel for the upper boundaries of Q and R as well as the lower. Since the diameters are the same, each corresponding pair of vertical edges can be replaced by the average of their respective heights. The new figures will still be closed and the sum of their areas will be unchanged. U will also be unchanged. With these modifications, Q becomes a translation of R, and hence $|U| = 4|R|$.

It follows that this optimized Δ is zero and hence the theorem is true. ▮

Corollary 1 Let $\widehat{Q}$ be a set of k convex polygons with equal horizontal diameters. Then

$$|Sor(\cup_{Q\in\widehat{Q}}Q)| \geq k \sum_{Q\in\widehat{Q}} |Q|.$$

Proof. Lemma 1 says that for $Q, R \in \widehat{Q}$, $|Q \oplus R| - |Q| - |R| \geq |R| + |Q|$. And since $|Sor(\cup_{Q\in\widehat{Q}}Q)| - \sum_{Q\in\widehat{Q}} |Q| = \frac{1}{2}\sum_{Q,R\in\widehat{Q};\ Q\neq R}(|Q \oplus R| - |Q| - |R|)$,

$$|Sor(\cup_{Q\in\widehat{Q}}Q)| - \sum_{Q\in\widehat{Q}} |Q| \geq \frac{1}{2} \sum_{Q,R\in\widehat{Q};\ Q\neq R} (|Q| + |R|) = (k-1) \sum_{Q\in\widehat{Q}} |Q|.$$

▮

Corollary 2. Suppose Q and R are convex polygons with horizontal diameters in the proportion $1 : \rho$. Then

$$|Q \oplus R| \geq |Q| + |R| + \rho|Q| + \frac{1}{\rho}|R|.$$

Proof. According to Lemma 1,

$$|(\sqrt{\rho}Q) \oplus (\frac{R}{\sqrt{\rho}})| \geq 2|\sqrt{\rho}Q| + 2|\frac{R}{\sqrt{\rho}}| =\geq 2\rho|Q| + \frac{2}{\rho}|R|.$$

The corresponding vector formulation for $|Q \oplus R|$ gives Q—Q cross products that sum exactly to $|Q|$, R—R cross products that sum exactly to $|R|$, and Q—R cross products that are exactly the same as the $\sqrt{\rho}Q$—$\frac{R}{\sqrt{\rho}}$ cross product terms.

Upon identifying equivalent portions, it follows that

$$|Q \oplus R| \geq |Q| + |R| + \rho|Q| + \frac{1}{\rho}|R|. \quad ▮$$

If ρ is unknown, ρ can be replaced with the value that minimizes $\rho|Q| + \frac{1}{\rho}|R|$.

Corollary B [Brunn-Minkowski].

$$|Q \oplus R| \geq |Q| + 2\sqrt{|Q| \cdot |R|} + |R|.$$

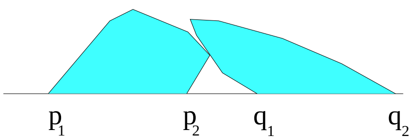
Proof: Solving for the minimum gives $\rho = \sqrt{|R|/|Q|}$. Substituting for ρ in Corollary 2 gives the Brunn-Minkowski inequality in 2 dimensions. ▮

Corollary 3. Let P and Q be convex open paths with the same orientation. Let P have endpoints p_1 and p_2, and let q_1 and q_2 terminate Q. Suppose that $\overline{p_1p_2}$ and $\overline{q_1q_2}$ are equal in length and parallel. Suppose that P has a total rotation that is bounded by π and likewise for Q. Finally, suppose that P and Q, when translated to have their endpoints all on one line, both lie on the same side of the line.

Then

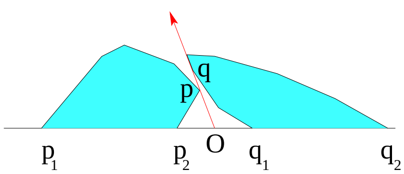
$$|ConvHll(P \oplus Q)| \geq 2(|ConvHll(P)| + |ConvHll(Q)|).$$

Proof. If P and Q have equal diameters with respect to support lines that are all parallel, then the claim follows from Lemma 1.



If not, P and Q can be translated to have their endpoints on a line ℓ so that the curves intersect at a point or segment, but have convex hulls that are otherwise disjoint, and the endpoint of P that is closest to Q is a positive distance from Q's closest endpoint. Let these two endpoints be p_2 and q_1.

For expositional convenience, let ℓ be the x-axis, and let P and Q be pulled apart so that p_2 is at the point $(-d, 0)$, q_1 is at $(d, 0)$, and let each curve be completely contained in, respectively, the second and the first quadrants. Imagine a ray that, in physical terms, is attached to a hinge located at the origin; its nominal direction is in an upward direction, but it is free to rotate in response to exerted torques.



Now let P and Q be slowly brought toward each other, with the origin kept as the midpoint between p_2 and q_1. Eventually one of the figures bumps into the ray and causes it to rotate. The process stops at moment the second curve makes contact with the ray. Let this fixed ray be $\vec{r}$. Let p be the point in $P \cap \vec{r}$ that is closest to the x-axis, and let q be the corresponding point for $Q \cap \vec{r}$. Let $\widehat{P}$ be P with the path from p to p_2 replaced by $\overline{p, O}$, where O is the origin. Define $\widehat{Q}$ analogously.

Let each curve have a counterclockwise rotation. By construction, $\widehat{P}$ and $\widehat{Q}$ have equal diameters with respect to support lines parallel to $\vec{r}$. By Lemma 1, $|ConvHll(\widehat{P} \oplus \widehat{Q})| \geq 2(|ConvHll(\widehat{P})| + |ConvHll(\widehat{Q})|)$.

Evidently, the curve $\widehat{P} \oplus \widehat{Q}$ will begin with a directed segment that is parallel and congruent to $\overrightarrow{Op}$ and will likewise end with a translation of $\overrightarrow{qO}$. Similarly, $P \oplus Q$ will begin with a copy of the path connecting p_2 and p along P, and end with a copy of the path that connects q and q_1 along Q.

Let a be the area of the region bounded by the x-axis, $\overline{Op}$, and the path connecting p_2 and p. Let b be the analogous area corresponding to the path connecting q and q_1.

By construction,

$$|ConvHll(\widehat{P} \oplus \widehat{Q})| = |ConvHll(P \oplus Q)| + a + b,$$

and by Lemma 1,

$$|ConvHll(\widehat{P} \oplus \widehat{Q})| \geq 2(|ConvHll(P)| + a + |ConvHll(Q)| + b).$$

Substituting for $|ConvHll(\widehat{P} \oplus \widehat{Q})|$ shows that

$$|ConvHll(P \oplus Q)| \geq 2(|ConvHll(P)| + |ConvHll(Q)|) + a + b.$$

∎

Several proofs will use simple reshaping procedures that are now formalized for expositional convenience.

Lemma 2 (The Slicing Lemma). Let Q and R be polygons. Let a straight segment split Q into two connected regions Q_1 and Q_2. Then

$$|Q \oplus R| \geq |Q_1 \oplus R| + |Q_2|.$$

Proof. Let $P = Q_1 \oplus R$. Let $\widehat{P}$ be the union of $ConvHll(P)$ and a translation of $ConvHll(Sor(Q_2))$ where the two figures have their corresponding copies of the edge that splits Q overlap perfectly. Then $|\widehat{P}| = |Q_1 \oplus R| + |Q_2|$. The lemma follows from noting that $|Sor(\partial\widehat{P})| \geq |\widehat{P}|$. ∎

A slight generalization is the following.

Lemma 3 (The Substitution Lemma). Let Q and R be oriented polygons. Let the consecutive vertices of Q be $q_1, q_2, \ldots, q_n$. Let P be an oriented polygon with consecutive vertices $p_1, p_2, \ldots, p_m$, and suppose that this sequence contains vertices of Q as a (not necessarily contiguous) subsequence but in the natural order as listed.

Then

$$|P \oplus R| \geq |Q \oplus R| + |Sor(P)| - |Sor(Q)|.$$

Proof. For a non-convex polygon X, let $\hat{X}$ denote the upper boundary of $Sor(X)$, and likewise let $\check{X}$ denote the lower. Since $|P \oplus R| = |Sor(P)| + |Sor(R)| + \hat{P} * \hat{R} + \check{P} * \check{R}$, and $|Q \oplus R| = |Sor(Q)| + |Sor(R)| + \hat{Q} * \hat{R} + \check{Q} * \check{R}$, the theorem will follow if we can show that $\hat{P} * \hat{R} + \check{P} * \check{R} \geq \hat{Q} * \hat{R} + \check{Q} * \check{R}$.

Of course area is isotropic; it is independent of the direction of the support lines that we use to define the upper and lower envelopes of each polygon. Consequently the Q—R cross products $\hat{Q} * \hat{R} + \check{Q} * \check{R} = |Q \oplus R| - |Sor(R)| - |Sor(Q)|$ comprise an isotropic sum.

Let e be a segment in $Sor(Q)$, and let g be the (parallel congruent) edge in Q that corresponds to e. Let $f_1, f_2, \ldots, f_k$ be consecutive segments of P that form the path that connects the endpoints of g and is in the same direction as g. Let the endpoints of g be α and β, and let the endpoints of f_1 be α and γ. Let X be the polygon formed by taking Q, removing g, and inserting f_1 and the (suitably) directed segment $\overline{\gamma\beta}$. We must show that the sum of the X—R cross products is at least as large as the sum of the Q—R cross products, since the replacements can be continued to substitute all of the edges of P for their counterparts in Q.

Let support lines parallel to f_1 be used to define the upper and lower boundaries of $Q \oplus R$ and $X \oplus R$. We are free to declare that f_1 is in either the upper or the lower boundary of $X \oplus R$, and select the one that contains $\overline{\gamma\beta}$. By convexity, it follows that g is also in the corresponding upper or lower boundary for $Q \oplus R$. Since all of the other edges in $Q \cup R$ must (or can) be in the corresponding boundary portions of both $Q \oplus R$ and $X \oplus R$, it follows that the cross products in $Q \oplus R$ must yield a sum that cannot exceed that for $X \oplus R$. ▮

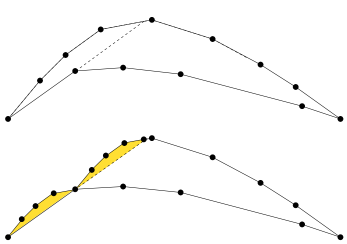
Lemma 4. Let P, Q, and R be convex polygons with $ConvHll(P) \supset Q$. Then

$$|P \oplus R| - |P| \geq |Q \oplus R| - |Q|.$$

Proof. It suffices to show that $|P \oplus R| - |P| - |R| \geq |Q \oplus R| - |Q| - |R|$. Let P be replaced by a rescaled polygon αP, where the new polygon is the minimum sized figure that is similar to P and still contains some translate of Q. Let Q be placed in a position of containment.

Since $|\alpha P \oplus R| - |\alpha P| - |R| = \alpha(|P \oplus R| - |P| - |R|)$, it suffices to prove the bound for the reduced P.

Let two or three points in $P \cap Q$ be selected that split both P and Q into two or three subpaths, where each subpath has a rotational change in its directed segments of at most π. Let P_i and Q_i be the resulting subpaths, for $i = 1, 2, 3$ or $i = 1, 2$. Suppose that for each i, P_i and Q_i have common endpoints and lie on the same side of the line through their endpoints.



Let the first edge of Q_i be extended to intersect P_i as shown. The extension slices off an initial portion of P_i, which is then reapportioned as to two similar subpaths as shown. To be specific, let the extended edge have endpoints r and t, and let s be the second vertex of Q_i. then the two sub paths are $\frac{|\overline{rs}|}{|\overline{rt}|}\rho_i$ and $\frac{|\overline{st}|}{|\overline{rt}|}\rho_i$, where ρ_i comprises the path along P_i from r to t. The net result is that the new edge arrangement now hits the second vertex in Q_i, and the procedure can now be applied to the next edge in Q_i.

Let $\widehat{P}_i$ be the final arrangement that intersects each vertex of Q_i, and let $\widehat{P}$ be the concatenation of the $\widehat{P}_i$. The Substitution Lemma ensures that

$$|\widehat{P} \oplus R| \geq |Q \oplus R| + |Sor(\widehat{P})| - |Sor(Q)|,$$

whence $|P \oplus R| - |P| - |R| \geq |Q \oplus R| - |Q| - |R|$, which gives the desired bound. ▮

Some applications of Lemma 1 plus a few additional observations will help bound the area that can accrue from the convex hull of two intersecting convex sets.

Definition. Let Q and R be convex polygons that intersect. Each connected component of $ConvHll(Q \cup R) \setminus (ConvHll(Q) \cup ConvHll(R))$ will be a *pocket* of $Q \cup R$. Each connected component of $(ConvHll(Q) \cup ConvHll(R)) \setminus (ConvHll(Q) \cap ConvHll(R))$ will be a will be a *finger* of $Q \cup R$.

Lemma 5 Let $\widehat{C}$ and $\widehat{D}$ be simple polygons with the same orientation. Suppose $C = \partial ConvHll(\widehat{C})$ and $D = \partial ConvHll(\widehat{D})$ are congruent k-gons with corresponding edges that are parallel. Let the edges of C be $\{e_i\}_{i=1}^{k}$. Let $\widehat{C}$ be partitioned into k edge-disjoint polygonal paths $\widehat{C}_1, \widehat{C}_2, \ldots, \widehat{C}_k$ where $\widehat{C}_i$ and e_i have the same endpoints, for $i = 1, 2, \ldots, k$. Set $C_i^+ = \partial ConvHll(\widehat{C}_i \cup e_i)$. Let $\widehat{D}_1, \widehat{D}_2, \ldots, \widehat{D}_k$ and $D_1^+, D_2^+, \ldots, D_k^+$ be defined analogously in terms of D and $\widehat{D}$.

Then

$$|\widehat{C} \oplus \widehat{D}| \geq 4|C| + 2\sum_i (|C_i^+| + |D_i^+|).$$

Proof. For $i = 1, 2, \ldots, k$, let $C_i = C_i^+ \setminus e_i$, provided $|C_i^+| > 0$, and otherwise let $C_i = e_i$. Let $D_1, D_2, \ldots, D_k$ be defined analogously. Let the vertices of C be $v_1, v_2, \ldots v_k$ and $e_i = \overline{v_i v_{i+1}}$. Let the vertices of D be $w_1, w_2, \ldots w_k$ and $f_i = \overline{w_i w_{i+1}}$. We can assume that e_i and f_i are parallel and of equal length.

Let $C^{-i} = (C \setminus e_i)$, and define D^{-i} analogously. For convenience, let all vertices, segments and subfigures retain the names of individual constituents used in their original definitions despite any subsequent processing.

Momentarily fix i.

Construct a (possibly degenerate) parallelogram that has side $\overline{v_i v_{i+1}}$, is contained within C^{-i}, and has as large an area as possible. Let the figure be $v_i c_1^+ c_2^+ v_{i+1}$. Let c_1^- and c_2^- be the additional vertices defining an analogous parallelogram for C_i.

Let c_1^+ and c_2^+ cut C^{-i} into the consecutive subpaths α_i^+, β_i^+ and γ_i^+. Let c_1^- and c_2^- cut C_i into the subpaths α_i^-, β_i^- and γ_i^-.

Let d_1^+ and d_2^+ define comparable locations on D^{-i}, and cut D^{-i} into the consecutive subpaths a_i^+, b_i^+ and g_i^+. Similarly, let d_1^- and d_2^- be the analogous vertices for D_i, and cut D_i into the subpaths a_i^-, b_i^- and g_i^-.

We can adjoin α_i^+ and γ_i^+; α_i^- and γ_i^-. Let the resulting figures be $C_{-||}^{-i}$ and $C_i^{-||}$. Let them share the adjoined endpoints that were originally named v_i and v_{i+1}. Evidently, $ConvHll(C_{-||}^{-i}) \supset C_i^{-||}$. Likewise, adjoin a_i^+ and g_i^+, and adjoin a_i^- and g_i^- to get, respectively, $D_{-||}^{-i}$ and $D_i^{-||}$.

The Brunn-Minkowski inequality ensures that

$$|C_{-||}^{-i} \oplus D_{-||}^{-i}| = 2|C_{-||}^{-i}| + 2|D_{-||}^{-i}|,$$

since the two figures are translations of each other. It also guarantees that

$$|C_{-||}^{-i} \oplus C_i^{-||}| - ||C_{-||}^{-i}| \geq 3|C_i^{-||}|.$$

Of course the same properties hold for $D_{-||}^{-i}$ and $D_i^{-||}$.

Let $\iota_i^+ = \beta_i^+ \oplus b_i^+$, and $\iota_i^- = \beta_i^- \oplus b_i^-$.

To account for twice the area of the parallelograms $v_i c_1^+ c_2^+ v_{i+1}$, $v_i c_1^- c_2^- v_{i+1}$, $v_i d_1^+ d_2^+ v_{i+1}$, $v_i d_1^- d_2^- v_{i+1}$, consider the following figure E_i.

Figure E_i is formed by adjoining $\alpha_i^+ \oplus \gamma_i^- \oplus a_i^+ \oplus g_i^-$, the segment connecting the endpoints of ι_i^+, $\gamma_i^+ \oplus \alpha_i^- \oplus g_i^+ \oplus a_i^-$, and the segment connecting the endpoints of ι_i^-. It is evident that the parallelogram with vertices defined by the located endpoints of of ι_i^+ and ι_i^- have at least twice the area of the four parallelograms. Let H_i be the infinite strip formed by two parallel lines connecting the endpoints of ι_i^+ with the corresponding endpoints of ι_i^-. We have already seen that the area of the pointset that is bounded by E_i and is exterior to H_i is at least $2|C_{-||}^{-i}| + 2|D_{-||}^{-i}| + 2|C_i^{-||}| + 2|D_i^{-||}|$.

We can now account for all of the pockets C_i and D_i. The construction logically replaces the i^{th} edge in $C \oplus D$ with the reverse sequenced ι_i^-. Let Ω be the resulting figure when such replacements are done for all i. Although Ω might not be simple, it follows that

$$|\Omega \oplus C_i^{-||} \oplus D_i^{-||}| - |Sor(\Omega)| \geq |E_i| \geq 2|D_i| + 2|C_i| - 2|ConvHll(\beta_i^-)| - 2|ConvHll(b_i^-)|.$$

By construction and the Substitution Lemma,

$$|\widehat{C} \oplus \widehat{D}| \geq |\Omega \oplus \cup_i (C_i^{-||} \cup D_i^{-||})|.$$

Hence,

$$\begin{aligned} |\widehat{C} \oplus \widehat{D}| &\geq |Sor(\Omega)| + \sum_i (|\Omega \oplus C_i^{-||} \oplus D_i^{-||}| - |Sor(\Omega)|) \\ &\geq |Sor(\Omega)| + 2\sum_i (|D_i^+| + |C_i^+| - |ConvHll(\beta_i^-)| - |ConvHll(b_i^-)|). \end{aligned}$$

From the Slicing Lemma, the definition of Ω, and Lemma 1, we have:

$$|Sor(\Omega)| \geq 4|C| + \sum_i |ConvHll(\iota_i^-)|.$$

From Corollary 3, it follows that

$$|ConvHll(\iota_i^-)| \geq 2|ConvHll(\beta_i^-)| + 2|ConvHll(b_i^-)|.$$

Consequently,

$$|\widehat{C} \oplus \widehat{D}| \geq 4|C| + 2\sum_i (|ConvHll(C_i)| + |ConvHll(D_i)|).$$

∎

Lemma 6 Let $\widehat{C}$ and $\widehat{D}$ be simple polygons with opposite orientations. Suppose $C = \partial ConvHll(\widehat{C})$ and $D = \partial ConvHll(D)$ are congruent k-gons with corresponding edges that are antiparallel. Then for some $\kappa \in [2\sqrt{2} - 2, \frac{\sqrt{13}-1}{3}]$,

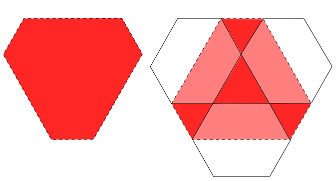
$$|\widehat{C} \oplus \widehat{D}| \geq 4|C| + (1+\kappa)(|C| - |\widehat{C}| + |D| - \widehat{D}|).$$

Moreover, the lemma cannot hold for any κ that exceeds $\frac{\sqrt{13}-1}{3} \approx .869$.

Proof. From the Brunn-Minkowski inequality,

$$\begin{aligned} |\widehat{C} \oplus \widehat{D}| &\geq |Sor(\widehat{C})| + 2\sqrt{|Sor(\widehat{C})| \cdot |Sor(\widehat{D})|} + |Sor(\widehat{D})| \\ &\geq 4|C| + (|\widehat{C}| - |C| + \widehat{D}| - |D|) + 2(\sqrt{|\widehat{C}| \cdot |\widehat{D}|}) - |C|) \\ &\geq 4|C| + (|\widehat{C}| - |C| + \widehat{D}| - |D|) + 2(|\widehat{C}| - |C| + \widehat{D}| - |D|)(\min_{0 \leq a \leq b \leq 1} \frac{\sqrt{(1+a)(1+b)} - 1}{a+b}), \end{aligned}$$

whence the lower bound for κ follows from the minimizing assignments $a = 0$, $b = 1$.



The upper bound for κ follows from the counterexample as drawn. $\widehat{C}$ is a convex equiangular hexagon whose sides have the alternating lengths $\frac{1+\sqrt{13}}{6}$ and $\frac{7+\sqrt{13}}{6}$. $\widehat{D}$ is actually degenerate, since it can be viewed as the union of four triangular cycles. It has three trapezoidal pockets as shown. The figure is illustrated with its edges reverse-sequenced so that the orientations of $\widehat{C}$ and $\widehat{D}$ are the same, but the shape of $\widehat{D}$ is effectively rotated by π. When this version of $\widehat{D}$ has its pockets reverse sequenced, the resulting figure is similar to $\widehat{C}$ and with parallel corresponding sides. The scaling factor is $1 : \frac{1+\sqrt{13}}{6}$. Thus, $\widehat{C} \oplus \widehat{D}$ is similar to C with a scaling factor of $\frac{7+\sqrt{13}}{6} : \frac{1+\sqrt{13}}{6}$. Inasmuch as the calculations are straightforward, they are omitted. ∎

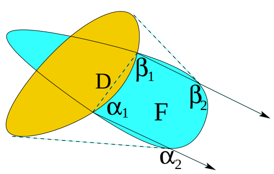
Lemma 7. Let C be a convex polygon. Suppose that $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are collections of, respectively c and d polygons whose convex hulls are translations of $ConvHll(C)$. Suppose that the polygons in $\widehat{\mathcal{C}}$ have clockwise orientations, and those in $\widehat{\mathcal{D}}$ have the opposite orientation. Then

$$|Sor(\cup_{Q\in\widehat{\mathcal{C}}}Q)| \geq (c+d)^2|C| + (c+\kappa\cdot d)\sum_{Q\in\widehat{\mathcal{C}}}(|C|-|Q|) + (d+\kappa\cdot c)\sum_{Q\in\widehat{\mathcal{D}}}(|C|-|Q|).$$

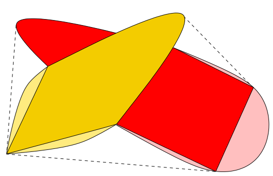
Proof. The inequality is a direct consequence of the derivation in Lemmas 5 and 6, and its decomposition into area-contributing cross products. ∎

The proofs in Section 3 use cases based on structural characteristics of intersecting cycles. We conclude the preliminaries by defining these characteristics and identifying a trivial property for one such structure.

Definitions Let Q and R be convex polygons that intersect. Let $\overline{U} = ConvexHull(Q) \cup ConvexHull(R)$, $\overline{I} = ConvexHull(Q) \cap ConvexHull(R)$, and $\overline{H} = ConvHll(\overline{U})$.

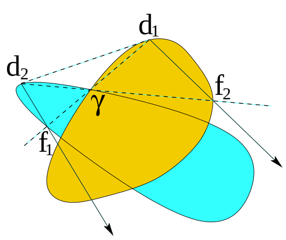


Let $\overline{F}$ be a finger of $\overline{U}$, and set $D = (\partial ConvHll(\overline{F})) \cap \overline{I}$, so that D is the best straight edge segment that slices $\overline{F}$ off of $\overline{U}$. Let D have endpoints α_1 and β_1. Let $\widehat{D} = \partial\overline{F} \cap \partial\overline{H}$, so that the polygonal curve $\widehat{D}$ comprises the common border between $\overline{F}$ and the exterior of $\overline{H}$. Let the endpoints of $\widehat{D}$ be α_2 and β_2, with the naming arranged so that the quadrilateral $\alpha_1\alpha_2\beta_2\beta_1$ is simple. The finger F is *slender* if the two infinite rays $\overrightarrow{\alpha_1\alpha_2}$ and $\overrightarrow{\beta_1\beta_2}$ intersect. Otherwise the finger is *fat*.



The Slicing Lemma will enable a finger F to be trimmed so that the modified F will have a boundary along the exterior of (the modified) $\overline{U}$ that comprises two or three edges. A finger with three edges along this external boundary will be said to have a *flat fingertip*, and a finger with just two such edges will be said to have a *pointed fingertip*.

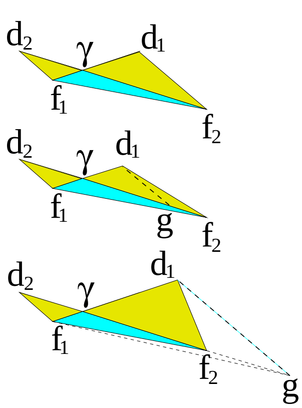
There are also two types of pockets.



Let $\overline{P}$ be a pocket of U. Let $D = (\partial\overline{P}) \cap \partial\overline{H}$, so that D is the straight edge segment that seals off the pocket inside of $\overline{H}$. Let d_1 and d_2 be the endpoints of D. Let γ be the pocket cusp $(\partial\overline{P}) \cap Q \cap R$, so that d_1, d_2 and γ are the strict extreme points of $\overline{P}$. Let f_1 be the last intersection point of $\overrightarrow{d_1\gamma}$ and $\overline{U}$, and likewise let f_2 be the analogous intersection point for $\overrightarrow{d_2\gamma}$ and $\overline{U}$. (To be more precise, f_1 is the unique point that is contained in $\overrightarrow{d_1\gamma} \cap \overline{U}$ and maximizes the length $|\overline{d_1 f_1}|$.) We say that $\overline{P}$ is a *deep* pocket if the infinite rays $\overrightarrow{d_1 f_2}$ and $\overrightarrow{d_2 f_1}$ intersect. If they do not, the pocket is *shallow*.

Note that convexity ensures that the implicitly defined triangles $\triangle d_2\gamma f_1$ and $\triangle d_1\gamma f_2$ are contained within their respective fingers, and $\triangle d_2\gamma d_1$ contains its pocket $\overline{P}$. As a consequence, the following triviality about quadrilaterals can be used to bound the area of a shallow pocket in terms of the areas of its neighboring fingers.

Lemma 8. Let $d_2 f_1 f_2 d_1$ be a convex quadrilateral as shown. Let its diagonals $\overline{d_2 f_2}$ and $\overline{d_1 f_1}$ intersect at the point γ.



1) If $\overline{d_2 f_1}$ is parallel to $\overline{d_1 f_2}$, then

$$|\triangle d_1 d_2\gamma| = \sqrt{|\triangle d_2 f_1\gamma|\cdot|\triangle d_1 f_2\gamma|}.$$

2) If $\angle d_2 f_1 f_2 + \angle d_1 f_2 f_1 < \pi$, then

$$|\triangle d_1 d_2\gamma| < \sqrt{|\triangle d_2 f_1\gamma|\cdot|\triangle d_1 f_2\gamma|}.$$

3) If $\angle d_2 f_1 f_2 + \angle d_1 f_2 f_1 > \pi$, then

$$|\triangle d_1 d_2\gamma| > \sqrt{|\triangle d_2 f_1\gamma|\cdot|\triangle d_1 f_2\gamma|}.$$

Proof.

1) The proof is a straightforward consequence of the similarity relationship: $\triangle d_2 f_1 \gamma \sim \triangle f_2 d_1 \gamma$.

2) Draw a line through d_1 that is parallel to $\overline{d_2 f_1}$. Let it intersect $\overline{d_2 f_2}$ at g. The proof follows from part 1 as applied to the quadrilateral $d_2 f_1 g d_1$.

3) Draw a line through d_1 that is parallel to $\overline{d_2 f_1}$. Extend $\overline{d_2 f_2}$ to intersect the line at g. The proof follows from part 1 as applied to the quadrilateral $d_2 f_1 g d_1$. ∎

3 More elaborate inequalities

The main result of this section is as follows.

Lemma 9. Let P be a polygon that is decomposed into a set $\mathcal{Z}$ of semisimple cycles.

Then

$$\begin{aligned} |Sor(\cup_{s\in\mathcal{Z}} s)| \quad \geq \quad & |ConvHll(P)| - |\cup_{s\in\mathcal{Z}} ConvHll(s)| \\ & + \sum_{s\in\mathcal{Z}} |ConvHll(s)| + 2 \sum_{\substack{\{s,t\}\subset\mathcal{Z} \\ s\neq t}} |ConvHll(s) \cap ConvHll(t)|. \end{aligned}$$

The proof is by induction. A stronger formulation of the base case is first established as a separate lemma.

Definition. Let v and w be located semisimple polygons. Define the function

$$CH^{\subset}(v,w) = \begin{cases} 1 & \text{if } v \subset ConvHll(w) \text{ and } v \text{ and } w \text{ have the same orientation;} \\ \kappa & \text{if } v \subset ConvHll(w) \text{ and } v \text{ and } w \text{ have oppopsite orientations;} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 10. Let Q and R be oriented semisimple cycles where $ConvHll(Q)$ and $ConvHll(R)$ intersect.

Then

$$\begin{aligned} |Q \oplus R| \quad \geq \quad & |ConvHll(Q \cup R)| + 3|ConvHll(Q) \cap ConvHll(R)| \\ & + (1 + CH^{\subset}(Q,R))(|ConvHll(Q)| - |Q|) + (1 + CH^{\subset}(R,Q))(|ConvHll(R)| - |R|). \end{aligned}$$

Proof. If the pointsets $ConvHll(Q)$ and $ConvHll(R)$ are the same, then the claim follows immediately from Lemmas 5 and 6.

So suppose that the convex hulls are different.

If one of the convex pointsets is a proper subset of the other, then the bound follows from Lemmas 4, 5, and 6, and the Substitution Lemma.

So suppose that neither convex hull is contained within the other. The Slicing Lemma shows that

$$|Q \oplus R| \geq |\partial ConvHll(Q) \oplus \partial ConvHll(R)| + (|ConvHll(Q)| - |Q| + |ConvHll(R)| - |R|),$$

where $\partial ConvHll(Q)$ and $\partial ConvHll(R)$ are given the same orientations as, respectively, Q and R.

Consequently, it suffices to suppose that neither convex hull contains the other, and to establish the Lemma for convex Q and R.

Let $\overline{I} = ConvHll(Q) \cap ConvHll(R)$. Let $\overline{Q}$ and $\overline{R}$ be the (unoriented) regions $ConvHll(Q)$, and $ConvHll(R)$. Let $Z = Q \oplus R$. The first two cases are straightforward.

Case 1. Both polygons have the same orientation. Let $P = \partial(\overline{Q} \cup \overline{R})$, and let $I = \partial(\overline{Q} \cap \overline{R})$. Both P and I should have the same orientation as Q. Thus, P is the outer boundary of $Q \cup R$, and I the inner. Let $\overline{P} = ConvHll(Sor(P))$. Now, $|\overline{P}| \geq |ConvHll(\overline{Q} \cup \overline{R})|$, and $ConvHll(I) = \overline{Q} \cap \overline{R}$. Thus, this case follows from the Brunn-Minkowski theorem: $|Sor(P) \oplus I| \geq |Sor(P)| + 2\sqrt{|Sor(P)| \cdot |I|} + |I| \geq |\overline{P}| + 3|\overline{Q} \cap \overline{R}|$, since $ConvHll(I) = \overline{I}$, and $\overline{I} \subset ConvHll(P)$.

Case 2. The polygons have opposite orientations and $\overline{Q} \cup \overline{R}$ has no deep pockets. Let $ConvHll(\overline{Q} \cup \overline{R})$ be decomposed into the following regions with non-intersecting interiors: the intersection $\overline{I} = \overline{Q} \cap \overline{R}$; the pockets $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_{2k}$;

and the fingers $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{2k}$, where $\overline{C}_{2i-1} \subset \overline{Q}$, $\overline{C}_{2i} \subset \overline{R}$, and $\overline{P}_i$ lies between the fingers $\overline{C}_i$ and $\overline{C}_{i+1}$. For convenience, we allow a finger to be a single point in degenerate cases, so that the desired alternation can always be achieved.

The Brunn-Minkowski inequality says that $|Q \oplus R| \geq |Q| + 2\sqrt{|Q| \cdot |R|} + |R|$.

By definition,

$$|Q| + |R| + \sum_i |\overline{P_i}| = |ConvHll(Q \cup R)| + |I|. \tag{1}$$

So it suffices to show that

$$2\sqrt{|Q| \cdot |R|} \geq 2|I| + \sum_i |\overline{P_i}|, \tag{2}$$

since the desired conclusion follows from combining the Brunn-Minkowski inequality, equality (1) and inequality (2) and cancelling $|Q| + |R| + 2\sqrt{|Q| \cdot |R|} + \sum_i |\overline{P_i}|$ term from both sides. Let

$$Lhs \equiv 2\sqrt{|Q| \cdot |R|}, \text{ and } Rhs \equiv 2|I| + \sum_i |\overline{P_i}|.$$

Substituting $|I| + \sum_i |C_{2i-1}|$ for $|Q|$, and $|I| + \sum_i |C_{2i}|$ for $|R|$ in the equation for Lhs and squaring gives

$$Lhs^2 = 4|I|^2 + 4|I| \sum_i |\overline{C_{2i-1}}| + 4|I| \sum_i |\overline{C_{2i}}| + 4(\sum_i |\overline{C_{2i-1}}|)(\sum_j |\overline{C_{2j}}|).$$

Rewriting gives

$$Lhs = 4|I|^2 + 4|I|A + 4B,$$

where

$$A = \sum_i (\sqrt{|\overline{C_{2i-1}}|})^2 + \sum_i (\sqrt{|\overline{C_{2i}}|})^2 \text{ and } B = (\sum_i (\sqrt{|\overline{C_{2i-1}}|})^2)(\sum_i (\sqrt{|\overline{C_{2i}}|})^2).$$

Lemma 8 ensures that $|\overline{P_i}| \leq \sqrt{|\overline{C_i}| \cdot |\overline{C_{i+1}}|}$. Substituting for $|\overline{P_i}|$ in the equation for Rhs and squaring gives $Rhs^2 \leq 4|I|^2 + 4|I| \sum_i \sqrt{|\overline{C_i}|} \cdot \sqrt{|\overline{C_{i+1}}|} + (\sum_i \sqrt{|\overline{C_i}|} \cdot \sqrt{|\overline{C_{i+1}}|})^2$. Equivalently,

$$Rhs \leq 4|I|^2 + 4|I|D_1 + 4|I|D_2 + (E + F)^2,$$

where $D_1 = \sum_i \sqrt{|\overline{C_{2i-1}}|} \cdot \sqrt{|\overline{C_{2i}}|}$, $D_2 = \sum_i \sqrt{|\overline{C_{2i}}|} \cdot \sqrt{|\overline{C_{2i+1}}|}$, $E = \sum_i \sqrt{|\overline{C_{2i}}|} \cdot \sqrt{|\overline{C_{2i+1}}|}$, and $F = \sum_i \sqrt{|\overline{C_{2i-1}}|} \cdot \sqrt{|\overline{C_{2i}}|}$.

The Cauchy-Schwartz inequality says that $\sqrt{B} \geq E$ and $\sqrt{B} \geq F$, so that $4B \geq (E + F)^2$. The law of cosines says that $A - 2D_1$ equals the square of the length of the vector $\vec{e}_1 - \vec{o}_1$, where $\vec{e}_1 = (\overline{C_2}, \overline{C_4}, \ldots \overline{C_{2k}})$, and $\vec{o}_1 = (\overline{C_1}, \overline{C_3}, \ldots \overline{C_{2k-1}})$. The analogous property holds for $A - 2D_2$, $\vec{e}_1$, and $\vec{o}_2 = (\overline{C_{2k}}, \overline{C_2}, \overline{C_4}, \ldots \overline{C_{2k-2}})$. Averaging ensures that $A - D_1 - D_2 = \frac{1}{2}||\vec{e}_1 - \vec{o}_1||^2 + \frac{1}{2}||\vec{e}_1 - \vec{o}_2||^2 \geq 0$. Consequently, $Lhs \geq Rhs$ as claimed.

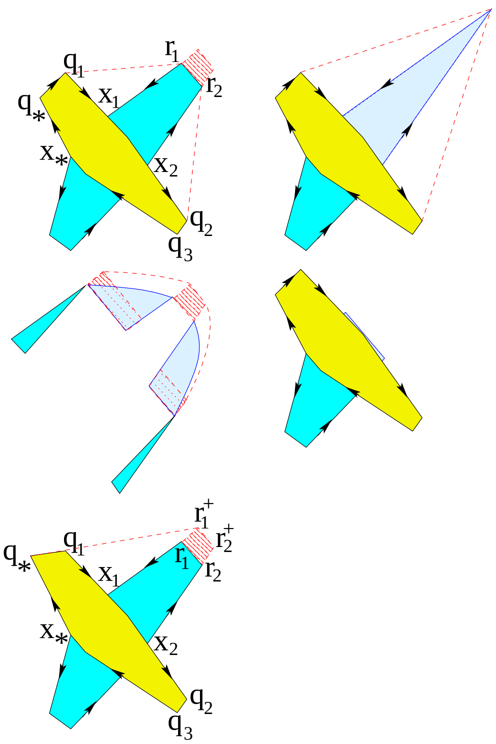
Case 3. The polygons have opposite orientations and deep pockets. A proof by contradiction simplifies the argument. So suppose P and Q are intersecting convex polygons that fail to satisfy the inequality. Our objective is to transform P and Q in ways that increase $\Delta \equiv |ConvHll(\overline{Q} \cup \overline{R})| + 3|\overline{I}| - |Q \oplus R|$, and eventually get regions without any the deep pockets.

In particular, the Slicing Lemma ensures that each pocket $\overline{P_i}$ can be replaced by $\widehat{P_i} \equiv ConvHll(P_i)$. Since Q and R are convex, $\widehat{P_i}$ must be a triangle. Its base equals $\partial(ConvHll(Q \cup R)) \cap \overline{P_i} \equiv \overline{q_i, r_i}$, where $q_i \in Q$ and $r_i \in R$. The opposing vertex is the point $x_i \equiv Q \cap R \cap P_i$. This finger slicing of Q and R reduces $|Q \oplus R|$ but leaves $|ConvHll(\overline{Q} \cup \overline{R})| + 3|\overline{Q} \cap \overline{R}|$ unchanged.

Similarly, we can modify each finger C_i so that the portion of its boundary that belongs to $\partial ConvHll(Q \cup R)$ is a segment. This simplification slices off fingertips of $ConvHll(Q \cup R)$, but the loss of area is easily seen to be the greatest for $|Q \oplus R|$. (Formally, $C_i \cup \partial ConvHll(Q \cup R)$ is a path ζ_i with endpoints p_i, q_i that belong to the neighboring pockets of C_i. Slicing the region C_i gives $(C_i \setminus ConvHll(\zeta_i)) \cup \overline{p_i q_i}$, if $\overline{p_i q_i}$ does not enter the interior of $\overline{I}$. If it does, the slicing cuts are taken as support lines to $\overline{I}$, which might split a finger into two fingers in need of tip slicing, or might result in the exposure of some common boundary along I and the newly resulting $\partial ConvHll(Q \cup R)$.)

This case concludes with two steps. The first transforms the sliced figure into a star-like shape with pointed fingertips. The second transformation eliminates the deep pockets.

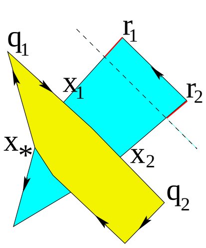
Let $U = Q \cup R$ be the original figure, and U_1 the trimmed figure with triangular pockets and sliced star-like fingers. The trimming almost produces the star-like figures we seek. The difficulty with the fingers is that they can terminate with flat fingertips rather than pointed tips.



Suppose that F is a slender finger in U_1. Let its terminating segment be $e_i(0) = \overrightarrow{r_2 r_1}$, and let the two connecting sides be $\overrightarrow{x_2 r_2}$ and $\overrightarrow{r_1 x_1}$ as shown. Define $e_i(t)$ to be the bounded segment formed from intersecting extensions of the two connecting sides with a line that is parallel to, and a distance t from $e_i(0)$. Let positive distance refer to lines that are shifted away from the figure. Let $l_i(t)$ and $r_i(t)$ be the extension of the respective sides that terminate at $e(t)$. Given these three modified edges, let $Q(t)$ and $R(t)$ be the modified polygons, $U(t) = Q(t) \cup R(t)$, $Z(t) = Sor(U(t))$, and let $H(t) = ConvHll(U(t))$. It is easy to see that the area change $|Z(t)| - |Z(0)|$ is the sum of the area of the trapezoid bounded by $e(0)$ and $e(t)$, and a term of the form at, for some constant a that is independent of t. Similarly, the change $|H(t)| - |H(0)|$ is the sum of the area for the same trapezoid and a comparable term bt. So we either increase t until $e(t)$ is a point or make t so negative that $e(t)$ hits a point of I. Either way, the reduction transforms a slender finger into zero, one, or two pointed fingers.

The bottom figure shows that intermediate transitions can occur. Here the finger growth from $\overline{r_1 r_2}$ to $\overline{r_1^+ r_2^+}$ has changed a neighboring pocket so that its base forms a straight line with the adjacent fingertip $\overline{q_* q_1}$. In this case, $\triangle q_* q_1 x_1$ is sliced away, which increases Δ. All improvement operations are applied to the new figure. Then the slender finger can continue to be changed in whatever direction increases Δ. (Actually, is not difficult to see that the direction will not change. Δ changes in a piecewise linear manner as a function of $|\overline{r_1^+ r_2^+}|$, and these events just turn out to accelerate the change. But this observation is not necessary.)

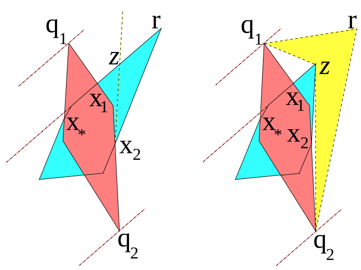
Now suppose that each slender finger has been so modified so that each slender finger is pointed. The next step is to eliminate the fat fingers.



Let F be a fat finger that, for specificity, belongs to R. Let F have the flat fingertip $\overrightarrow{r_2 r_1}$, and suppose that F lies between the pockets $\triangle q_1 x_1 r_1$ and $\triangle r_2 x_2 q_2$. Let ℓ be a cut parallel to $\overline{r_1 r_2}$.

Let the upper and lower boundaries of figures Q R and Z, be determined by support lines that are parallel to $\overline{r_1 x_1}$. Let $\overrightarrow{r_1 x_1}$ belong to the boundary portions that have the same upper/lower designation as $\overrightarrow{q_1 x_1}$. Consequently, the four directed segments $\overrightarrow{x_2 r_2}$, $\overrightarrow{q_1 x_1}$, $\overrightarrow{x_2 q_2}$, and $\overrightarrow{r_1 x_1}$ all have the same designation. Hence, we can slice off the fingertip of F with a line parallel to $\overline{r_1 r_2}$. The line should intersect F and be a support line of ∂I. The three sides $\overrightarrow{r_1 r_2}$, $\overrightarrow{r_1 x_1}$, and $\overrightarrow{x_2 r_2}$ have their lengths reduced but their directions do not change. There is an area loss for $|R|$ that is reflected in the change of the R—R crossproducts. The R—Q cross products decrease in value that is at least as large as the area loss for the two pockets $\triangle q_1 x_1 r_1$ and $\triangle q_2 x_2 r_2$, since they all have their sides in the same boundary portion of $Q \oplus R$. By choosing ℓ to be a support line of Q, F will either split into one or two slender fingers or be trimmed completely away.

Hence all fat fingers can be eliminated without decreasing Δ. The last step is to show that when all fingers are slender, all deep pockets can be eliminated.



Let $x_2 r x_1$ and $x_1 q_1 x_*$ be the envelopes of two fingers with the intervening pocket $\triangle r x_1 q_1$, and suppose that the pocket is deep. Let the point z be located on $\overline{rx_1}$. Replace $\overline{x_1 r}$ with $\overline{x_1 z}$, and replace $\overline{x_2 r}$ with $\overline{x_2 z}$. Suppose, for specificity, that $\overrightarrow{rz}$ represents the direction of the subsegment $\overline{rz}$ in the oriented polygon R. The point z should be as close as possible to x_1 subject to the constraints that $\overline{x_2 z}$ not enter the interior of I and no edge of Q or R have a directed slope that lies between the directed slopes of $\overrightarrow{x_2 r}$ and $\overrightarrow{x_2 z}$. (Equal slopes are permitted.) These trimmings change the area of the convex hull by $\frac{1}{2}|\overrightarrow{q_1 q_2} \times \overrightarrow{rz}|$. They also reshape $Q \oplus R$. In particular, $|Q \oplus R|$ is decreased by $\frac{1}{2}|\overrightarrow{x_2 r} \times \overrightarrow{rz}| + |\vec{p} \times \overrightarrow{rz}|$, where $\vec{p}$ is the sum of the directed edges in Q and R with orientations between the directions $\overrightarrow{x_2 r}$ and $\overrightarrow{rx_1}$.

Extend $\overrightarrow{rx_1}$ to intersect $\overrightarrow{x_* q_1}$. Draw a ray $\vec{\ell}$ that emanates from q_2 and is parallel to $\overrightarrow{rx_1}$. Draw a line ℓ_1 through q_1 that is parallel to $\overrightarrow{rx_1}$. The convexity of R ensures that and ℓ_1 and R lie in opposite halfplanes as defined by the line through $\overline{rx_1}$. Since all fingertips are pointed, it follows that all of Q lies to one side of ℓ_1. Let σ be the portion of Q that goes backward from q_1 to x_* and continues up to its first intersection with $\vec{\ell}$ (which need not be q_2).

By the convexity of Q, all segments of σ have directions that lie between the orientations of $\vec{\ell}$ and $\overline{x_* q_1}$. Since the path runs from $\vec{\ell}$ to the parallel line ℓ_1, the area loss for $|Q \oplus R|$ is at least $\frac{1}{2}|\overrightarrow{x_2 r} \times \overrightarrow{rz}| + |\overline{q_1 q_2} \times \overrightarrow{rz}|$, which exceeds the loss of $\frac{1}{2}(|\overrightarrow{x_2 r} \times \overrightarrow{rz}| + |\overline{q_1 q_2} \times \overrightarrow{rz}|)$ for the convex hull H.

The net result is that Δ does not decrease, and the pocket $\triangle q x_1 z$ is less deep. The process can be repeated until either the pocket vanishes altogether or is no longer deep. The reason that this step might have to be repeated is that the cut, as illustrated, might be along a support line of Q, and the next cut would then start at a different vertex of Q.

Eventually each deep pocket will be eliminated and Case 2 will become applicable. Since Δ never decreases, the Lemma is established. ∎

Now the proof of Lemma 9 can be completed.

Lemma 9. Let P be a polygon that is decomposed into a set $\mathcal{Z}$ of semisimple cycles.

Then

$$
\begin{aligned}
|Sor(\cup_{s \in \mathcal{Z}} s)| \geq\ & |ConvHll(P)| - |\cup_{s \in \mathcal{Z}} ConvHll(s)| \\
& + \sum_{s \in \mathcal{Z}} |ConvHll(s)| + 2 \sum_{\substack{\{s,t\} \subset \mathcal{Z} \\ s \neq t}} |ConvHll(s) \cap ConvHll(t)|.
\end{aligned} \tag{3}
$$

Proof. Let n be the number of elements in $\mathcal{Z}$. It is clear that the lemma is true for $n = 1$.

For $n = 2$, let $\mathcal{Z} = \{Q, R\}$. Lemma 10 ensures that

$$|Q \oplus R| \geq |ConvHll(Q \cup R)| + 3|ConvHll(Q) \cap ConvHll(R)|. \tag{4}$$

Since

$$0 = |ConvHll(Q)| + |ConvHll(R)| - |ConvHll(Q) \cup ConvHll(R)| - |ConvHll(Q) \cap ConvHll(R)|, \tag{5}$$

and $P = Q \cup R$, the conclusion follows from adding inequality 4 and equation 5.

The proof is completed by inductive contradiction. Suppose the claim is not true. Let P be a polygon with decomposition $\mathcal{Z}$ where the bound fails to hold and the number of elements in $\mathcal{Z}$ is as small as possible. Let this count be n.

We can assume that each set is convex since replacing each cycle by the boundary of its convex hull will not change the right-hand side of equation 3, but will, according to the Substitution Lemma, decrease the left-hand side.

Suppose that some pair of distinct cycles $u, v \in \mathcal{Z}$ have the same orientation and happen to intersect. Let $z_1 = \partial(ConvHll(u) \cup ConvHll(v))$, and $z_2 = Sor(\partial(ConvHll(u) \cap ConvHll(v)))$. Let $\mathcal{Z}_0 = \mathcal{Z} \setminus \{u, v\}$, and $\mathcal{Z}_1 = \{z_1\} \cup \mathcal{Z}_0$. Then the edge collection in $\mathcal{Z}$ is the same as the edges in $\{\mathcal{Z}_1\} \cup \{z_2\}$.

By the inductive minimality assumption, Lemma 9 holds for $\mathcal{Z}_1$, and therefore

$$\begin{aligned}|Sor(\cup_{s\in\mathcal{Z}_1} s)| - \sum_{s\in\mathcal{Z}_1} |ConvHll(s)| \quad \geq \quad & |ConvHll(P)| - |\cup_{s\in\mathcal{Z}_1} ConvHll(s)| \\ & +2 \sum_{\substack{\{s,t\}\subset\mathcal{Z}_0 \\ s\neq t}} |ConvHll(s) \cap ConvHll(t)| \\ & +2 \sum_{\{s\}\subset\mathcal{Z}_0} |ConvHll(s) \cap ConvHll(z_1)|.\end{aligned}$$

Now,

$$\begin{aligned}|z_2 \oplus Sor(\cup_{s\in\mathcal{Z}_1} s)| - |z_2| - |Sor(\cup_{s\in\mathcal{Z}_1} s)| \quad & \geq \quad \sum_{s\in\mathcal{Z}_1} (|z_2 \oplus s| - |z_2| - |s|) \\ & \geq \quad \sum_{s\in\mathcal{Z}_1} 2\sqrt{|z_2|\cdot|s|} \\ & \geq \quad \sum_{s\in\mathcal{Z}_1} 2|ConvHll(z_2) \cap ConvHll(s)|\end{aligned}$$

Combining these two inequalities, with the observation that for all $s \in Z_0$:

$$\begin{aligned}2|ConvHll(s) \cap ConvHll(z_1)| \quad \geq \quad & 2|ConvHll(s) \cap ConvHll(u)| + 2|ConvHll(s) \cap ConvHll(v)| \\ & - 2|ConvHll(s) \cap ConvHll(z_2)|\end{aligned}$$

gives:

$$\begin{aligned}|Sor(\cup_{s\in\mathcal{Z}} s)| - \sum_{s\in\mathcal{Z}_1} |ConvHll(s)| - |z_2| \quad \geq \quad & |ConvHll(P)| - |\cup_{s\in\mathcal{Z}_1} ConvHll(s)| \\ & +2 \sum_{\substack{\{s,t\}\subset\mathcal{Z} \\ s\neq t}} |ConvHll(s) \cap ConvHll(t)|, \qquad (6)\end{aligned}$$

where we used the fact that $|ConvHll(z_2) \cap ConvHll(z_1)| = |ConvHll(u) \cap ConvHll(v)|$. Evidently,

$$|ConvHll(z_1)| - |u| - |v| + |z_2| = |ConvHll(z_1) \setminus (ConvHll(u) \cup ConvHll(v))|.$$

Combining the left-hand side of this equality with inequality 6, and combining the corresponding sets in a subadditive manner within $-|\cup_{s\in\mathcal{Z}_1} ConvHll(s)|$ on the right gives:

$$\begin{aligned}|Sor(\cup_{s\in\mathcal{Z}} s)| - \sum_{s\in\mathcal{Z}} |ConvHll(s)| \quad \geq \quad & |ConvHll(P)| - |\cup_{s\in\mathcal{Z}} ConvHll(s)| \\ & +2 \sum_{\substack{\{s,t\}\subset\mathcal{Z} \\ s\neq t}} |ConvHll(s) \cap ConvHll(t)|.\end{aligned}$$

So in the refuting decomposition $\mathcal{Z}$, every pair of intersecting cycles must have opposite orientations.

Let $G = (V, E)$ and Cy be, respectively, an undirected graph and a 1-to-1 mapping of V onto the elements of $\mathcal{Z}$ where for $u \neq v \in V$, $(u, v) \in E$ if and only if $Cy(v) \cup Cy(u) \neq \emptyset$. In this case, we say that the pair (G, Cy) is an *incidence representation* for $\mathcal{Z}$. For any graph H with vertex set V_H in the domain of Cy, let $Cy(H) = \cup_{v \in V_H} Cy(v)$.

Now suppose that G is not a tree, so that it contains a cycle. Let the $G_1 = (V_1, E_1)$ be a cycle in G with the minimum number of vertices. Evidently G_1 comprises an even closed loop of four or more intersecting cycles with alternating orientations. It is easy to see that these cycles can be replaced by two oppositely oriented cycles that each traverse the full chain. This restructuring actually increases the right-hand side of inequality 3 while preserving the left. Since the number of cycles has decreased, the bound must again hold as a consequence of the minimality assumption.

The only remaining possibility is where G is a tree, and all intersecting cycles have opposite orientations. In this case, we say that the incidence representation for $\mathcal{Z}$ is a *tree of alternating cycles*.

Let G be represented by the notation $T = (V, E_T)$.

We must show that

$$
\begin{aligned}
|Sor(\cup_{s \in \mathcal{Z}} s)| - \sum_{s \in \mathcal{Z}} |ConvHll(s)| \quad \geq \quad & |ConvHll(P)| - |\cup_{s \in \mathcal{Z}} ConvHll(s)| \\
& + 2 \sum_{(s,t) \in E_T} |ConvHll(s) \cap ConvHll(t)|. \qquad (7)
\end{aligned}
$$

Definition. Let $\mathcal{Z}$ have an incidence representation (T, Cy) that is a tree of alternating convex cycles. Suppose that w is a leaf of T and that v is its parent. We say that w is a *minimal leaf* if only one connected component of $Cy(v) \setminus ConvHll(Cy(w))$ has intersections with other cycles.

Of course, some vertices may have leaves but no minimal leaf. However, it is easy to see that as consequence of the convexity of each cycle, the deepest leaf in T must have at least one minimal leaf.

Let w be a minimal leaf of T, and v be its parent. If v has no parent in T, let $u \neq w$ be a child of v, and let T be restructured so that u is the root of T. Thus, we can assume that v has the parent u. Let T_{-w} be the tree T with w removed.

Definition. Let $\mathcal{Z}$ have an incidence representation (T, Cy) that is a tree of alternating cycles. Let w be a minimal leaf of T, and v be the parent of w. Suppose that v has the parent u in T. We say that ζ is a *proxy* for $Cy(w) \cup Cy(v)$ in T if the following hold.

1. ζ is a located convex polygon that has the same orientation as $Cy(v)$.
2. For $z \in V_T \setminus \{v, w\}$, $ConvHll(Cy(z)) \cap ConvHll(Cy(v)) \subset ConvHll(Cy(z)) \cap ConvHll(\zeta)$.
3. Let $T_\nu = (V_\nu, E_\nu)$ be the tree T_{-w} with v replaced by ν, where $Cy(\nu) = \zeta$. Then the pockets defined by $Cy(T_\nu)$ contain the pocket portions defined by $Cy(T)$ that are exterior to $ConvHll(Cy(v \cup w))$. Formally,
$$\big(ConvHll(Cy(T)) \setminus \cup_{s \in V_T} ConvHll(Cy(s))\big) \setminus ConvHll(Cy(v \cup w))$$
$$\subset ConvHll(Cy(T_\nu)) \setminus \cup_{s \in V_\nu} ConvHll(Cy(s))$$
4. Let ζ comprise the directed located segments $\zeta_1, \zeta_2, \ldots, \zeta_k$. Then a subset of the directed located segments in $Cy(v) \cup Cy(w)$ can be apportioned into edge-wise disjoint subpaths $\mathcal{Z}_1, \mathcal{Z}_2, \ldots \mathcal{Z}_k, \ldots$, where $\mathcal{Z}_i$ and ζ_i have the same endpoints and equivalent directions, for $i = 1, 2, \ldots, k$.

Intuitively, a proxy can be used to account for all of the area that can be attributed to interactions between $Cy(v) \cup Cy(w)$ and the rest of the figure. These interactions appear in two settings: as cross products in $|Sor(Cy(T))|$ that involve exactly one segment in $Cy(v) \cup Cy(w)$, and as subregions that belong to intersections or pockets formed from interactions between $ConvHll(Cy(v) \cup Cy(w))$ and the rest of the figure.

Note that 4 implies

5. $ConvHll(\zeta) \subset ConvHll(Cy(w) \cup Cy(v))$.

Now suppose that ζ is a proxy for $Cy(w) \cup Cycle(v)$ in T. By property 2, $Cy(v) \cap Cy(x) \subset ConvHll(\zeta)$ for any $x \in V \setminus w$. Consequently, T_ν is connected. Technically, a cycle $Cy(x)$ might have an intersection with ζ despite having no intersection with $Cy(v)$, since ζ can have located segments in regions where $Cy(v)$ might have none. No matter; the formula in equation 7 is fixed. The additional intersections are not represented in the area formula we seek to prove.

In any case, the induction hypothesis applies to the located cycles (imperfectly) represented by T_ν to ensure that

$$\begin{aligned}|Sor(Cy(T_\nu))| &\geq |ConvHll(Cy(T_\nu))| - |\cup_{s\in V_\nu} ConvHll(Cy(s))| \\ &\quad + \sum_{s\in V_\nu} |ConvHll(Cy(s))| + 2\sum_{(s,t)\in E_\nu} |ConvHll(s)\cap ConvHll(t)|.\end{aligned}$$

Moreover, proxy property 4 and the Substitution Lemma guarantees that

$$|Sor(Cy(T))| \geq |Sor(Cy(T_\nu)) + |Sor(Cy(v\cup w))| - |\zeta|,$$

since the convexity of ζ ensures that $|\zeta| = |Sor(\zeta)|$.

Lemma 10 shows that

$$|Sor(Cy(v\cup w))| \geq |ConvHll(Cy(v\cup w))| + 3|ConvHll(Cy(v)) \cap ConvHll(Cy(w))|.$$

These three inequalities can be combined to give:

$$\begin{aligned}|Sor(Cy(T))| &\geq |ConvHll(Cy(T_\nu))| - |\cup_{s\in V_\nu} ConvHll(Cy(s))| \\ &\quad + \sum_{s\in V_\nu} |ConvHll(Cy(s))| - |\zeta| + 2\sum_{(s,t)\in E_\nu} |ConvHll(s)\cap ConvHll(t)| \\ &\quad + |ConvHll(Cy(v\cup w))| + 3|ConvHll(Cy(v))\cap ConvHll(Cy(w))| \\ &\geq |ConvHll(Cy(T_\nu))| - |\cup_{s\in V_\nu} ConvHll(Cy(s))| \\ &\quad + \sum_{s\in V_T} |ConvHll(Cy(s))| - |ConvHll(Cy(v))\cup ConvHll(Cy(w))| \\ &\quad + 2\sum_{(s,t)\in E_\nu} |ConvHll(s)\cap ConvHll(t)| \\ &\quad + |ConvHll(Cy(v\cup w))| + 2|ConvHll(Cy(v))\cap ConvHll(Cy(w))| \end{aligned} \tag{8}$$

Proxy property 2 guarantees that

$$ConvHll(Cy(z)) \cap ConvHll(\zeta) \geq ConvHll(Cy(z)) \cap ConvHll(Cy(v)),$$

so the intersections with ζ can be replaced by intersections with $Cy(v)$. Hence

$$\begin{aligned}|Sor(Cy(T))| &\geq |ConvHll(Cy(T_\nu))| - |\cup_{s\in V_\nu} ConvHll(Cy(s))| \\ &\quad + \sum_{s\in V_T} |ConvHll(Cy(s))| - |ConvHll(Cy(v))\cup ConvHll(Cy(w))| \\ &\quad + 2\sum_{(s,t)\in E_T} |ConvHll(s)\cap ConvHll(t)| + |ConvHll(Cy(v\cup w))|.\end{aligned} \tag{9}$$

According to property 3,

$$\begin{aligned}&ConvHll(Cy(T_\nu)) \setminus \cup_{s\in V_\nu} ConvHll(Cy(s)) \\ &\qquad \supset (ConvHll(Cy(T)) \setminus \cup_{s\in V_T} ConvHll(Cy(s))) \setminus ConvHll(Cy(v\cup w)).\end{aligned}$$

It follows that

$$\begin{aligned}|ConvHll(Cy(T_\nu))| - |\cup_{s\in V_\nu} ConvHll(Cy(s))| &\geq |ConvHll(Cy(T))| - |\cup_{s\in V_T} ConvHll(Cy(s))| \\ &\quad - (|ConvHll(Cy(v\cup w))| \\ &\quad - |ConvHll(Cy(v))\cup ConvHll(Cy(w))|).\end{aligned} \tag{10}$$

Combining inequalities 9 and 10 shows that

$$|Sor(Cy(T))| \geq |ConvHll(Cy(T))| - |\cup_{s\in V_T} ConvHll(Cy(s))| + \sum_{s\in V_T} |ConvHll(Cy(s))| + 2\sum_{(s,t)\in E_T} |ConvHll(s)\cap ConvHll(t)|,$$

which establishes Lemma 9 for $Cy(T)$.

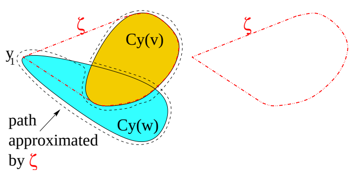
Consequently, it suffices to present proxies for $Cy(v)\cup Cy(w)$ whenever possible, and to offer remedies for the instances where none is to be found.

There are several cases. To distinguish among them, let $G = ConvHll(Cy(T_{-w})\setminus Cy(v))$, and let the line segments $\overline{x_1y_1}$ and $\overline{x_2y_2}$ comprise $(\partial ConvHll(G\cup Cy(w)))\setminus(G\cup ConvHll(Cy(w)))$, with y_1 and y_2 belonging to $Cy(w)$.

Case 000: y_1 and y_2 do not exist. This occurs only if $Cy(w)$ lies in the interior of $ConvHll(Cy(T_{-w}))$. In this circumstance the induction step is trivial because the inclusion of $Cy(w)$ does not increase the convex hull of the resulting figure. A suitable proxy is $Cy(v)$.

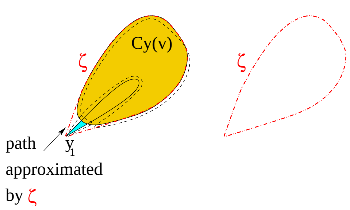
Case 00: $x_1 = x_2$. This cases is postponed to the end.

Case 0: $Cy(v)$ intersects both $\overline{x_1y_1}$ and $\overline{x_2y_2}$.
This case is trivial because $Cy(u)$ only interacts with $Cy(v)$. A suitable proxy is $Cy(v)$.

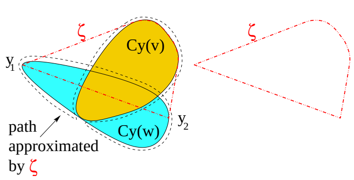


Case 1: $Cy(v)$ intersects exactly one of the two segments $\overline{x_1y_1}$, $\overline{x_2y_2}$.
For specificity, let $\overline{x_1y_1}$ not intersect $Cy(v)$. Then a suitable proxy is defined by $\zeta = \partial ConvHll(Cy(v)\cup y_1)$ with an orientation consistent with that of $Cy(v)$.

Case 2: $Cy(v)$ intersects neither $\overline{x_1y_1}$ nor $\overline{x_2y_2}$.
Let V_u be the component of $Cy(v)\setminus Cy(w)$ that intersects $Cy(u)$, and let its endpoints be v_1 and v_2. Let W_{ext} be the portion of $Cy(w)$ that terminates at endpoints y_1 and y_2 and does not intersect V_u. There are several subcases.



Case 2a: $y_1 = y_2$. In this circumstance, $W_{ext} = y_1$. Then a suitable proxy is $\zeta = \partial ConvHll(Cy(v)\cup y_1)$ with an orientation consistent with that of $Cy(v)$.



Case 2b: $Cy(v)$ intersects W_{ext}. One or more intersections points can be used to satisfy proxy criterion 4 and construct a proxy as follows. Let V_0 be the component of $Cy(v)\setminus\overline{y_1y_2}$ that intersects $Cy(u)$. Then a suitable proxy is defined by

$$\zeta = \partial ConvHll(V_0\cup\overline{y_1y_2}),$$

with an orientation consistent with that of $Cy(v)$.

Case 2c: $Cy(v)$ does not intersect W_{ext}. Let W_{ext+} be the component of $Cy(w)\setminus Cy(v)$ that contains W_{ext}, and let its endpoints be z_1 and z_2, with z_1, y_1, y_2, z_2 lying in sequential order on W_{ext+}.

Let

$$\Delta = |ConvHll(P)| - |\cup_{v\in V_T} ConvHll(Cy(v))| + \sum_{v\in V_T} |ConvHll(Cy(v))|$$

$$+2\sum_{(v,y)\in E_T}|ConvHll(Cy(v))\cap ConvHll(Cy(y))|-|Sor(\cup_v Cy(v))|.$$

Finger analysis will be used to modify $Cy(T)$ in ways that do not cause Δ to decrease, and that transform the figure into cases where proxies or the equivalent can be found. Although the expression for Δ might seem a little complicated, it will suffice to modify P in ways that are easy to analyze.

Case 2cI: $\overline{y_1y_2}$ intersects $Cy(v)$. The Slicing Lemma ensures that $Cy(w)$ can be trimmed with a segment that begins at y_1, is a support line to $Cy(v)$, and terminates on W_{ext}. The segment is used to replace the boundary path that connects its endpoints. In this circumstance, the Slicing Lemma ensures that Δ will not decrease. The resulting figure will satisfy the conditions of Case 2b.

Case 2c$\overline{I}$: $\overline{y_1y_2}$ does not intersect $Cy(v)$. As in Case 2cI, the Slicing Lemma is used to trim, via path replacement, $Cy(T)$ along $\overline{y_1y_2}$. Evidently Δ cannot decrease. The new figure satisfies $W_{ext}=\overline{y_1y_2}$.

Let the located segments $Cy(T)$ be viewed as a graph that includes all intersection points of segments as vertices, and all induced subsegments as directed edges. The figure is connected. It is also the union of oriented cycles. Hence it is strongly connected. Consequently, y_1, y_2, x_1 and x_2 must lie on some closed path of directed subsegments. Evidently y_1 and y_2 must be on the path in consecutive order. While the orientation of this order is not important, it is possible that x_1 and x_2 complete the sequence in one of the two orderings y_1, y_2, x_1, x_2 or y_1, y_2, x_2, x_1.

It is desirable to use path replacement to trim W_{ext+} along $\overline{y_1z_1}$, but the replacement segment might intersect $Cy(v)$. Accordingly, let δ_1 lie on the path connecting y_1 and z_1 and be as close to z_1 as possible, subject to the constraint that $\overline{y_1\delta_1}$ not have any points in the interior of $ConvHll(Cy(v))$. Let δ_2 be defined analogously for y_2 and z_2. $Cy(w)$ can be reshaped the have segments $\overline{y_1y_2}$, $\overline{y_1\delta_1}$ and $\overline{y_2\delta_2}$. The Slicing Lemma again ensures that Δ cannot decrease.

Case 2c$\overline{I}$S: The finger $\delta_1y_1y_2\delta_2$ is slender. In this case, the argument used to prove Lemma 10 applies either to grow the finger to the point where $y_1=y_2$, which is Case 2a, or to trim it with slices parallel to the base up to the point where $\overline{y_1y_2}$ intersects $Cy(v)$, which is Case 2b.

Case 2c$\overline{I}$F: The finger $\delta_1y_1y_2\delta_2$ is fat. The subcases are handled as follows.

Case 2c$\overline{I}$F1: The order of vertices comprises an orientation of the sequence δ_1, y_1, y_2, δ_2, x_2, x_1. Let Σ be the polygon formed by connecting these vertices in the order listed.

Since Σ is simple, $\partial ConvHll(\Sigma)$ is, in a sense, a virtual proxy. The edges are sums of disjoint collections of edges in $Cy(T)$. Moreover, if slicing is used to trim edges of $Cy(w)$ in ways that only affect pockets and areas that do not intersect other cycles, then the area loss for $Sor(Cy(T))$ must be at least as large as that for $ConvHll(\Sigma)$.

To be specific, let ℓ be the parallel to $\overline{y_1y_2}$, that is a support line for $Cy(v)$, and is as close to $\overline{y_1y_2}$ as possible. Let ℓ intersect $\overline{y_1\delta_1}$ at ϵ_1 and $\overline{y_2\delta_2}$ at ϵ_2. The figure is sliced along $\overline{\epsilon_1\epsilon_2}$. Let B be a triangle with sides parallel to $\overline{y_1y_2}$, $\overline{y_1\delta_1}$ and $\overline{y_2\delta_2}$ and let the side parallel to $\overline{y_1y_2}$ have length $|\overline{y_1y_2}|-|\overline{\epsilon_1\epsilon_2}|$.

Let S be the located segment pieces (which comprise a non-simple polygon) in $Cy(T)$ that can be grouped as disjoint collections of subsegments to produce the located collection Σ (that is simple). Let (the non-simple) S_{-B} and (the simple) Σ_{-B} be the analogous located edge collections that result from the slicing of the fat finger by ℓ.

Then $S_{-B}\oplus B\equiv Sor(S)$, and $|ConvHll(\Sigma_{-B})\oplus B|\geq|ConvHll(\Sigma)|$.

By the Substitution Lemma,

$$\begin{aligned}|Sor(S_{-B})\oplus B|-|Sor(S_{-B})| &\geq |Sor(\Sigma_{-B})\oplus B|-|Sor(\Sigma_{-B})|\\ &\geq |ConvHll(\Sigma_{-B})\oplus B|-|ConvHll(\Sigma_{-B})|\\ &\geq |ConvHll(\Sigma)|-|ConvHll(\Sigma_{-B})|,\end{aligned}$$

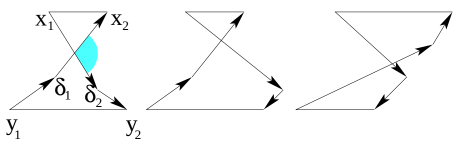
where the last inequality is due to the previous observation about $|ConvHll(\Sigma_{-B})\oplus B|$. So the area loss due to trimming B from $|Sor(Cy(T))|$ is at least as large as the loss due to the trimming of Σ, which includes all pocket losses and finger reductions that result from the actual changes in $|ConvHll(Cy(T))|$.

Thus, the trimming eliminates the fat finger to produce Case 2b.

Case 2c$\overline{I}$F2: The order is δ_1, y_1, y_2, δ_2, x_1, x_2. Connecting the vertices in this order gives a self-intersecting polygon. Let ℓ be parallel to $\overline{y_1y_2}$, intersect $\overline{y_1\delta_1}$, and be a support line for $Cy(v)$. Let ℓ intersect $\overline{y_1\delta_1}$ and $\overline{y_2\delta_2}$ at ϵ_1 and ϵ_2, respectively.

In this case, $Cy(w)$ is trimmed along ℓ. The three finger sides of $Cy(w)$ undergo a reduction in length (of course, one of the reductions could be zero, if the fat finger has parallel sides), but the directions do not change. Let s_1 be the vector trimmed from $\overline{y_1\delta_1}$, s_2 the vector trimmed from $\overline{y_2\delta_2}$, and s_3 the vector trimmed from $\overline{y_1y_2}$. Since all cross product terms for $|Sor(Cy(T))|$ are positive and the slicing just shortens some of the edges, the area loss for $|Sor(Cy(T))|$ is the sum of the loss induced in $|Cy(w)|$ plus the sum of the appropriate s_1, s_2, s_3—$Cy(T_{-w})$ cross products. The trimming reduces the area in $|ConvHll(Cy(T))|$ by at most $\frac{1}{2}(|s_1 \times \overrightarrow{y_1x_1}| + |s_2 \times \overrightarrow{y_2x_2}| + |s_1 \times \overrightarrow{y_1y_2}| + |s_2 \times \overrightarrow{y_2x_2}|)$, which equals $\frac{1}{2}(|s_1 \times \overrightarrow{x_1y_2}| + |s_2 \times \overrightarrow{x_2y_1}|)$.

The Substitution Lemma ensures that the actual loss for $|Sor(Cy(T))|$ is at least $\frac{|s_1 \times s_2|}{2}$ plus the appropriate cross product contributions between s_1, s_2, s_3 and the six-segment polygon $\delta_1, y_1, y_2, \delta_2, x_1, x_2$. The Substitution Lemma also ensures that this latter loss is bounded by appropriate cross products between s_1, s_2, s_3 and the quadrilateral $y_1y_2x_1x_2$.



For expositional convenience, let the edge orientations be as shown, so that $\overline{y_1\delta_1}$ is directed as $\overrightarrow{y_1\delta_1}$.

It suffices to show that the vectors $\overrightarrow{y_1\delta_1}$, $\overrightarrow{y_1x_1}$, $\overrightarrow{\delta_2y_2}$, $\overrightarrow{x_1y_2}$ all belong to a range of directions that is bounded by π, since then a suitable direction to define upper and lower boundaries will include the cross products $\frac{1}{2}(|s_1 \times \overrightarrow{x_1y_2}| + |s_2 \times \overrightarrow{x_2y_1}|)$ in the decrease for $|Sor(Cy(T))|$.

Evidently the (clockwise) rotation of $\overrightarrow{\delta_1x_2}$ to $\overrightarrow{x_1x_2}$ to $\overrightarrow{x_1\delta_2}$ is less than π because the angle of rotation is the supplemental to the angle subtending the base of the triangle with base $\overline{x_1x_2}$ and opposing vertex located at the intersection point $\overline{\delta_1x_2} \cap \overline{x_1\delta_2}$. Let R represent this range of directions. If the directions of $\overrightarrow{y_1\delta_1}$ and $\overrightarrow{\delta_2y_2}$ lie within R we are done. So suppose that $\overrightarrow{y_1\delta_1}$ does not. The definition of this case requires that $\overrightarrow{y_1\delta_1}$ lie within the convex boundary $y_1x_1x_2y_2$, Hence the rotation from $\overrightarrow{y_1\delta_1}$ to $\overrightarrow{\delta_1x_2}$ to $\overrightarrow{x_1\delta_2}$ is less than $\pi - \angle y_1x_1\delta_2 < \pi$. So even if $\overrightarrow{y_1\delta_1}$ extends the range of R, it is still less that π. If both $\overrightarrow{y_1\delta_1}$ and $\overrightarrow{\delta_2y_2}$ extend the range, then the resulting range is from $\overrightarrow{y_1\delta_1}$ to $\overrightarrow{x_1x_2}$ to $\overrightarrow{\delta_2y_2}$. But this range must be at most π because these segments belong to the actual finger that is being processed, and it is fat. The trimming cannot decrease Δ, and yields a figure where Case 2b is applicable.

Case 000: For completeness, we note that x_1 can equal x_2, in which case the finger is fat and the polygon formed from connecting y_1, y_2 and x_1 is simple. ∎

4 The main bound

At this point, all of the infrastructure needed to prove Theorem 2 has been established.

Proof of Theorem 2. Let the cycles in $\mathcal{Z}$ be partitioned into equivalence classes where $s \equiv t$ if $ConvHll(s) = ConvHll(t)$. Let containment define a partial order on the set of equivalence classes: define $s < t$ to mean $s \subset ConvHll(t)$. Let $\mathcal{Z}_1$ contain one representative from each containment-based maximal equivalence class.

Since $\cup_{s\in\mathcal{Z}_1} ConvHll(s)$ contains every cycle, it contains P. From Lemma 9, we have:

$$\begin{aligned}|Sor(\partial \cup_{s\in\mathcal{Z}_1} ConvHll(s))| \quad \geq \quad & |ConvHll(P)| - |\cup_{s\in\mathcal{Z}_1} ConvHll(s)| \\ & + \sum_{s\in\mathcal{Z}_1} |ConvHll(s)| + 2 \sum_{\substack{\{s,t\}\subset\mathcal{Z}_1 \\ s\neq t}} |ConvHll(s) \cap ConvHll(t)|.\end{aligned}$$

It follows that

$$|Sor(\cup_{s\in\mathcal{Z}_1} s)| \quad \geq \quad \sum_{s\in\mathcal{Z}_1} (|Sor(s)| - |ConvHll(s)|) + |ConvHll(P)| - |\cup_{s\in\mathcal{Z}_1} ConvHll(s)|$$

$$+\sum_{s\in\mathcal{Z}_1}|ConvHll(s)|+2\sum_{\substack{\{s,t\}\subset\mathcal{Z}_1\\ s\neq t}}|ConvHll(s)\cap ConvHll(t)|. \tag{11}$$

Let $a=|ConvHll(P)|-|\cup_{s\in\mathcal{Z}_1}ConvHll(s)|$. Combining the definition of a with inequality 11 shows that

$$|Sor(\cup_{s\in\mathcal{Z}_1}s)| \;\;\geq\;\; a+2\sum_{\substack{\{s,t\}\subset\mathcal{Z}_1\\ s\neq t}}|ConvHll(s)\cap ConvHll(t)|+\sum_{s\in\mathcal{Z}_1}|Sor(s)|.$$

So

$$|Sor(\cup_{s\in\mathcal{Z}_1}s)|-\sum_{s\in\mathcal{Z}_1}|Sor(s)| \;\;\geq\;\; a+2\sum_{\substack{\{s,t\}\subset\mathcal{Z}_1\\ s\neq t}}|ConvHll(s)\cap ConvHll(t)|. \tag{12}$$

The bilinear form that evaluates $|Sor(\cup_{s\in\mathcal{Z}_1}s)|-\sum_{s\in\mathcal{Z}_1}|Sor(s)|$ contains no terms for area within an individual cycle in $\mathcal{Z}_1$, that is, all products are formed from pairs of segments that belong to different cycles in $\mathcal{Z}_1$.

Let, for $s\in\mathcal{Z}_1$, $\mathcal{Z}_s^+=\{t\in\mathcal{Z}:\ s$ and t have the same orientation and $ConvHll(s)=ConvHll(t)\}$. Let w_s^+ equal the number of elements in $\mathcal{Z}_s^+$. Define $\mathcal{Z}_s^-$ and w_s^- analogously for the orientation opposite to that of s, and let $w_s^\pm=w_s^++w_s^-$. Let $\mathcal{Z}_2=\cup_{s\in\mathcal{Z}_1}\cup_{t\in\mathcal{Z}_s^+\cup\mathcal{Z}_s^-}\{t\}$.

From Lemma 7, the fact that $\mathcal{Z}_1$ contains no equivalent cycles, and since the form $|Sor(\cup_{s\in\mathcal{Z}_1}s)|-\sum_{s\in\mathcal{Z}_1}|Sor(s)|$ contains no cross products with edges from the same cycle, it follows that

$$\begin{aligned}|Sor(\cup_{s\in\mathcal{Z}_2}s)| \;\;\geq\;\; & |Sor(\cup_{s\in\mathcal{Z}_1}s)|-\sum_{s\in\mathcal{Z}_1}|Sor(s)|+\sum_{s\in\mathcal{Z}_1}(w_s^++w_s^-)^2|ConvHll(s)|\\ &+\sum_{s\in\mathcal{Z}_1}\sum_{t\in\mathcal{Z}_s^+}(w_s^++\kappa\cdot w_s^-)(|ConvHll(t)|-|t|)\\ &+\sum_{s\in\mathcal{Z}_1}\sum_{t\in\mathcal{Z}_s^-}(w_s^-+\kappa\cdot w_s^+)(|ConvHll(t)|-|t|)\\ &+\sum_{s\in\mathcal{Z}_1}\sum_{\substack{t\in\mathcal{Z}_1\\ t\neq s}}(w_s^\pm w_t^\pm-1)|ConvHll(s)\cap ConvHll(t)|.\end{aligned} \tag{13}$$

Combining inequalities 12 and 13 gives:

$$\begin{aligned}|Sor(\cup_{s\in\mathcal{Z}_2}s)| \;\;\geq\;\; & a+\sum_{s\in\mathcal{Z}_1}(w_s^++w_s^-)^2|ConvHll(s)|+\sum_{s\in\mathcal{Z}_1}\sum_{\substack{t\in\mathcal{Z}_1\\ t\neq s}}w_s^\pm w_t^\pm|ConvHll(s)\cap ConvHll(t)|\\ &+\sum_{s\in\mathcal{Z}_1}\sum_{t\in\mathcal{Z}_s^+}(w_s^++\kappa\cdot w_s^-)(|ConvHll(t)|-|t|)\\ &+\sum_{s\in\mathcal{Z}_1}\sum_{t\in\mathcal{Z}_s^-}(w_s^-+\kappa\cdot w_s^+)(|ConvHll(t)|-|t|),\end{aligned}$$

whence

$$|Sor(\cup_{s\in\mathcal{Z}_2}s)| \;\;\geq\;\; a+\sum_{s\in\mathcal{Z}_1}\sum_{t\in\mathcal{Z}_1}w_s^\pm w_t^\pm|ConvHll(s)\cap ConvHll(t)|$$

$$+ \sum_{s \in \mathcal{Z}_1} \sum_{t \in \mathcal{Z}_s^+} (w_s^+ + \kappa \cdot w_s^-)(|ConvHll(t)| - |t|)$$
$$+ \sum_{s \in \mathcal{Z}_1} \sum_{t \in \mathcal{Z}_s^-} (w_s^- + \kappa \cdot w_s^+)(|ConvHll(t)| - |t|). \tag{14}$$

Evidently, the right-hand side of inequality 14 is equivalent to

$$\int_{\mathbf{x} \in ConvHll(P_2)} w_2(\mathbf{x})^2 d\mathbf{x} + \sum_{C \in \mathcal{Z}_2} W_2(C)(|ConvHll(C)| - |C|), \tag{15}$$

where $P_2 = \cup_{s \in \mathcal{Z}_2} s$, and w_2 and W_2 are defined as in Theorem 2 for the polygon P_2.

To complete the argument, it suffices to include, via the Sor operation, each of the remaining cycles in $\mathcal{Z} \setminus \mathcal{Z}_2$ one equivalence class at a time. Lemma 7 gives sufficient area to permit the bilinearity of area as a function of the edges plus the Brunn-Minkowski inequality to account for all of the remaining terms in Theorem 2. ▮

5 Conclusions and Extensions

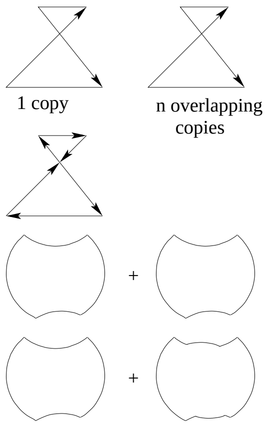
Our purpose was to formulate a notion of area causality where the consequences of overlapping boundaries and non-convexity are quantified in terms of the convex hull operation, the footprint sizes of variously defined subregions, and the winding of boundary curves around them. From this perspective, containment and membership in the convex hull are the geometric properties that characterize various subregions of P. It is not difficult to show that Theorem 2 will be false if any term is increased by a fixed multiplicative factor, apart from the uncertainty built into the definition of κ. In this sense, at least, the bounds are reasonably tight.

Of course, the bounds can be improved by including additional characteristics of the polygon. The Brunn-Minkowski inequality, for example, gives minimum estimates for area contributions that are caused by the interaction among any collection of edges or cycles, even if their convex hulls have no overlap. In addition, the notion of an unsigned winding number can be extended to some areas that are included in the convex hull of the polygon P. For example, let x be a point inside a pocket of P that is sealed off by a line ℓ on its convex hull. The formulas presented do not take account of the possibility that among all continuous curves that connect x to the unbounded component of $R^2 \setminus P$, and which do not cross ℓ, the minimum number of crossings of P might exceed 2.

There are also some immediate extensions to Theorem 2. It should be noted that in Lemma 5, an actual pocket of, say, $\widehat{C}$ can be replaced by its in-place Sor, provided this convexification leaves it contained within $ConvHll(\widehat{C})$. This strengthening is a departure from the naive use of containment and convex hulls, but is helpful for broadening the range of figures where the area estimate is tight.

Similarly, it is possible to extend the analysis that accounted for area lying outside of the individual cycles. Theorem 2 was based on an initial decomposition that created Z_1, which contained one representative from each maximal cycle class. While the approach accounts fairly accurately for instances of overlapping intersections of cycles, it fails to account for an analogous overlap of pockets formed from chains of cycles that do not belong to $\mathcal{Z}_1$. An enhanced approach would repeat the top-level decomposition at subsequent rounds and thereby give a much better result for, say, a figure composed of multiple superimposed copies of a polygon.

The following illustrate some of the cases where Theorem 2 and its extensions are strong. In all cases, the multiple figures have the same convex hull, and overlap perfectly.



Theorem 2 is strong enough to account for most of the area from a single copy of the non-simple quadrilateral, but multiple copies need an analysis based on the repeated selection of maximal semisimple cycle representatives.

Because the pockets use edges that have the same orientation (and a single flip convexifies each pocket), Theorem 2 is exact in this case, and also applies to multiple copies.

Likewise, Theorem 2 is tight if both cycles have the same orientation. The idea is that a flip of each pocket creates a circle. Multiple copies are again handled correctly.

In this case, the area for each pocket has to be computed from the Sor extension to get an exact result. Tightness requires the multiple regions to have the same orientation, the same convex hulls and the same figures when Sor-ed.

It is also interesting to examine a case where the bound cannot be exact, but ought to be very good. Let P comprise two superimposed cycles: a half-disk, and a folded-over circle that has no area. Let the two semicircular curves overlap perfectly. Let $h = \frac{\pi r^2}{2}$ be the area of the half-disk. Then Theorem 2 gives an area estimate of $4h + 2h = 6h$. The actual area of the Sor of the two figures (as formalized below for piecewise smooth curves) is $4h + h/2 + 2r^2 + h/2 = (5 + \frac{4}{\pi})h$, which is rather close. All previous bounds yield an estimate of h. Of course, "opening up" the folded-over disk permits a substantial improvement for the Banchoff and Pohl bound, which returns an estimate of $5h$ for the case of a half-disk superimposed on top of a disk. The Brunn-Minkowski inequality improves the estimate to $(3 + 2\sqrt{2})h \approx 5.8h$. On the other hand, the estimate provided by Theorem 2 decreases to 5h because the convex hull enhancement fails to reveal any hidden area, and the one-way containment of the two regions does not impart enough information to "deduce" that the cycles have equal directional diameters.

In retrospect, we see that the notion of causality is, in part, an illusion. If cause is to be apportioned locally, then we expect aggregate contributions to be additive. The Brunn-Minkowski inequality, plus simple convexification says that

$$\left(\sum_{s \in \mathcal{Z}} \sqrt{2|ConvHll(s)| - |s|}\right)^2 \leq |Sor(P)|,$$

which ignores overlap, and casts a blind eye to pockets formed from different cycles. Theorem 2 can beat this bound only under special circumstances. One case is when large amounts of area comprise external pockets as in the case of a self-intersecting quadrilateral. For this particular example, Theorem 2 is sharper, as are the bounds of Pach, and Böröczky, Bárány, Makai and Pach. The other circumstance is when different cycles have equal directional diameters, whence the formulations of Lemma 1 and its consequences can be used instead of the Brunn-Minkowski inequality. Of course, there must be sufficient information for Theorem 2 to "know" that the diameters are the same, and our containment primitive is adequate when cycles have identically superimposed convex hulls.

On the other hand, inequalities are typically a tradeoff between expressiveness and precision, and the bounds in this paper give a semantics for identifying certain kinds of area contributions that had heretofore been unquantified.

In closing, we note that the Sor has a natural interpretation when passing to the limit. Let $\mathcal{L}(\ )$ measure arclength and be defined on countable unions of rectifiable curves. Let $\gamma_i(s)$ be a family of oriented piecewise smooth simple cycles, where s is the arclength parameter for γ_i, so that $\mathcal{L}(\{\gamma_i(s) : 0 \leq s \leq x\}) = \min(x, \mathcal{L}(\gamma_i))$. Let $\theta_i(s)$ be the angle that the outward normal to γ_i at $\gamma_i(s)$ makes with respect to a ray running horizontally from $\gamma_i(s)$ to the right. Let $f(s)$ be the unique convex curve with with an outward normal whose comparably defined angle $\theta_f(s)$ satisfies:

$$\mathcal{L}(\{f(s) : 0 \leq \theta_f(s) \leq \alpha\}) = \sum_i \mathcal{L}(\{\gamma_i(s) : 0 \leq \theta_i(s) \leq \alpha\})$$

for $0 \leq \alpha \leq 2\pi$. Let A_f be the area of the region bounded by f.

Then the natural extension is to replace $|Sor(P)|$ with A_f in the strong formulation for Theorem 2.

Theorem 3. Suppose γ is a closed oriented rectifiable curve in the plane, and Z is a decomposition of γ into oriented semisimple cycles. Let A_f be as defined above and W and w be as in Theorem 1. Then

$$\int_{\mathbf{x} \in ConvHll(\gamma)} w(\mathbf{x})^2 d\mathbf{x} + \sum_{C \in \mathcal{Z}} W(C)(|ConvHll(C)| - |C|) \leq A_f. \quad \blacksquare$$

Alternatively, a simpler version can be stated in terms of arclength.

Theorem 4. Let γ be a closed curve with finite arclength ℓ. Let $Q_1, Q_2, \ldots$ be the layered reoriented decomposition of γ as described at the end of Section 1.2. Let $Q_0 = R^2$. Let $w(\mathbf{x}) = \max_i\{i :\ \mathbf{x} \in ConvHll(Q_i)\}$. For a cycle Q_j, let $W(Q_j) = \max_i\{i :\ Q_j \subset ConvHll(Q_i)\}$. Then

$$\int w(\mathbf{x})^2 d\mathbf{x} + \sum_{i\geq 1} W(Q_i)(|ConvHll(Q_i)| - |Q_i|) \leq \frac{1}{4}|Sor(Q) \oplus Sor(Q^{Rev})| \leq \frac{\ell^2}{4\pi},$$

where Q^{Rev} is Q with its orientation reversed. It follows trivially from Lemma 1 that a half-sized scaled down version of $Sor(Q) \oplus Sor(Q^{Rev})$ has the greatest area among all rearrangements of the infinitesimal segments comprising Q, when rotations are not allowed and the orientation of each element is ignored. $\blacksquare$

This formulation gives all external pockets of γ a weighting of 2 or more, rather than a weighting of one as Theorem 2 does for the external pockets that have edges with opposite orientations. Similarly, this version is more elegant since it eliminates the possibility of weak area contributions as formalized in Lemma 6.

There are also some extensions that are worth posing as conjectures and open questions.

Let F be a finite collection of segments located in the R^2 For any $x \in R^2$, let $N(x)$ be the smallest number of segments intersected by any line through x. Given a simple polygon C, let C^{+int} denote the region bounded by C.

Conjecture. Theorems 2, 3 and 4 still hold if $w(\mathbf{x})$ is replaced by $\frac{1}{2}N(\mathbf{x})$.

Conjecture. Theorem 1 still holds if $w(\mathbf{x})$ is replaced by $\frac{1}{2}N(\mathbf{x})$, the second term is dropped, and P is redefined to comprise a finite collection of segments in the plane.

Open questions.

What is the correct value for κ?

Suppose, for simplicity, that all cycles have the same orientation.

Can $\sum_{C\in\mathcal{Z}} W(C)(|ConvHll(C)| - |C|)$ in Theorem 2 be replaced by $\sum_{C\in\mathcal{Z}} \int_{\mathbf{x}\in ConvHll(C)\setminus C^{+int}} w(\mathbf{x})$?

Can $\sum_{C\in\mathcal{Z}} W(C)(|ConvHll(C)| - |C|)$ in Theorem 2 be replaced by $\sum_{C\in\mathcal{Z}} \int_{\mathbf{x}\in ConvHll(C)\setminus C^{+int}} \frac{1}{2}N(\mathbf{x})$?

Some related questions can be found in [8].

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