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Upstart Puzzles

Seesaw Gold

[**Figure:** Seesaw of seven meters with single support attached at the middle, leftmost point meter has no weight, next meter has a one kilogram gold weight in the center, next meter has no weight, fulcrum in the middle, one kilogram, no weight no weight. [All weights are gold]

(**caption**) If torque is more on the left side than on the right, all weights will move one space to the left. If a weight is at the leftmost position and the seesaw tilts left, the weight falls off the seesaw. At that point, of course, that weight no longer exercises torque. All this works symmetrically on the right side. How does the situation in this figure evolve?

Consider a seesaw of length six meters. The fulcrum is at the three meter mark. There is a gold weight of one kilogram at the one meter mark and one at the four meter mark. We represent this as follows $_1_f1_$ where the $_$ represents no weight, the 1 represents a weight of one, and the f represents the fulcrum which is just a point. (See the figure on top.)

Warm-up: Once we let the seesaw tilt, what will happen?

Solution to warm-up:

Because the torque on the left side is greater than the torque on the right side, the seesaw tilts towards the left. The configuration will evolve as follows:

$_1_f1_$ initial configuration

$1_1f_$ all weights move one to the left (skipping over the fulcrum meter)

$_1_f_$ the leftmost weight falls off the edge

$1_f_$

$_f_$ the second weight falls off and seesaw regains balance

We can ask many questions about such a system.

Question: Can you create a seesaw with twice the weight on the right side than on the left but where all the gold weights fall to the left?

Solution: Here is one of many possibilities:

$1_f2_$ initial configuration more torque on the left

$_2f_$ left weight falls off but right weight goes to left side

Question:

Can you find a starting configuration of length six meters that will start by tilting one way and then tilt the other before losing all the weights?

Solution: Here is one solution.

1 _ f _ 1 _

which evolves as follows

1 _ f _ 1 _ initial

_ f _ 1 _ tilts to the left but loses that first weight

_ f _ 1 _ tilts to the right

_ f _ 1 _ keeps tilting

_ f _ the second weight falls off and seesaw regains balance

Question: On a seesaw of length 14, create a configuration such the seesaw tilts to the left, then to the right, then to the left again.

Solution:

1 _ 1 _ f _ 1 _ initial

_ 1 _ f _ 1 _ left weight falls off

_ 1 _ f _ 1 _ moves towards the right

_ 1 _ f _ 1 _ moves towards the right

_ 1 f _ right weight falls off, so now seesaw will tilt to left

Now let's consider a game. One player is called Right and one is called Left. Each player wants as much weight in gold as possible to fall to his or her side. The seesaw is fixed at some length. Left places weights and then Right places weights. This is called the *Left-first* version of the game. Because Right has an advantage in the Left-first game, Left has one advantage: Left receives any weights on the seesaw if the seesaw ever reaches balance (as much torque on the left as on the right).

Question: Suppose Left has weights 1, 2, 3 as does Right. Suppose the board is only of length 7 (three to the left, three to the right, and one central meter where the fulcrum is), then can Left guarantee to win?

Solution: Surprisingly, Left will win with 321fxyz where x, y, and z represent any values that Right puts on its part of the board. Here's why. First, if Right sets things up as 321f123, then the two sides have the same torque, so the seesaw will be in balance in which case Left gets all the weights. This implies that Right should allow the left side to have more torque at the beginning but then make it so the right side has more torque thereafter. The best possible such configuration is 321f132. Let's see what happens in that case. After one time unit, the configuration becomes 211f32_, because the leftmost 1 on the right side moves to the left. But then the torque on the left becomes $6 + 2 + 1 = 9$ and on the right $3 + 4 = 7$, so the board will stay tilted to the left thereafter. Because all weights fall to the left, let's call this a *shutout*.

Question: Now consider a board of length 9. Suppose Left has weights 3, 2, 1 in this configuration 321_fxyzw. Where can Right put weights 1, 2, 3 in order to make as much weight as possible to fall to the right and avoid a balanced seesaw?

Solution:

321 _ f _ 132 initial

21 _ f132 _ move to the left, 3 falls to the left, torque is 11 on left ($8+3$) and 13 on right

_ 21 _ f _ 132 move to the right

_ 21 f _ 13 move to the right

___2f1___1 move to the right
___f21___ move to the right

The weights that fall to the right are all the initial rightside weights and the 2 and 1 from the left side.

End of solution

Question: Can Right always win when the weights are 1, 2, 3 and the board is of length 9?

Solution: No. Left can win with this: 231_fxyzw. Right cannot choose 231_f_123 because then Right would win once and then lose thereafter. Right cannot choose 231_f_132, because that would be in balance. Right could however choose: 231_f_231

Left torque: $8 + 9 + 2 = 19$; Right torque: $4 + 9 + 4 = 17$.

Go left to

31__f231__ Left torque: 13 Right Torque = 11

Go left to

1__2f31__ Left torque: 6. Right torque: 6

This would continue and be a shutout for Left. If Right did not use all three of the rightmost meters, the same configuration would again lead to a shutout.

Upstart 1: In the Left-first game, given a board length, weights for Left, and weights for Right, try to find an algorithm that determines a winning configuration for Left (meaning Right never gets more weight in total than Left) if there is one regardless of what Right does. If there is none, show how Right can win no matter how Left places its pieces. Do the same for shutouts.

We call the variation of the game in which Left places the first weight, then Right places a weight and so on until both sides run out *Alternating*.

Upstart 2: Answer the previous Upstart for the Alternating configuration.

All are invited to submit their solutions to upstartpuzzles@cacm.acm.org; solutions to upstarts and discussion will be posted at <http://cs.nyu.edu/cs/faculty/shasha/papers/cacmpuzzles.html>

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