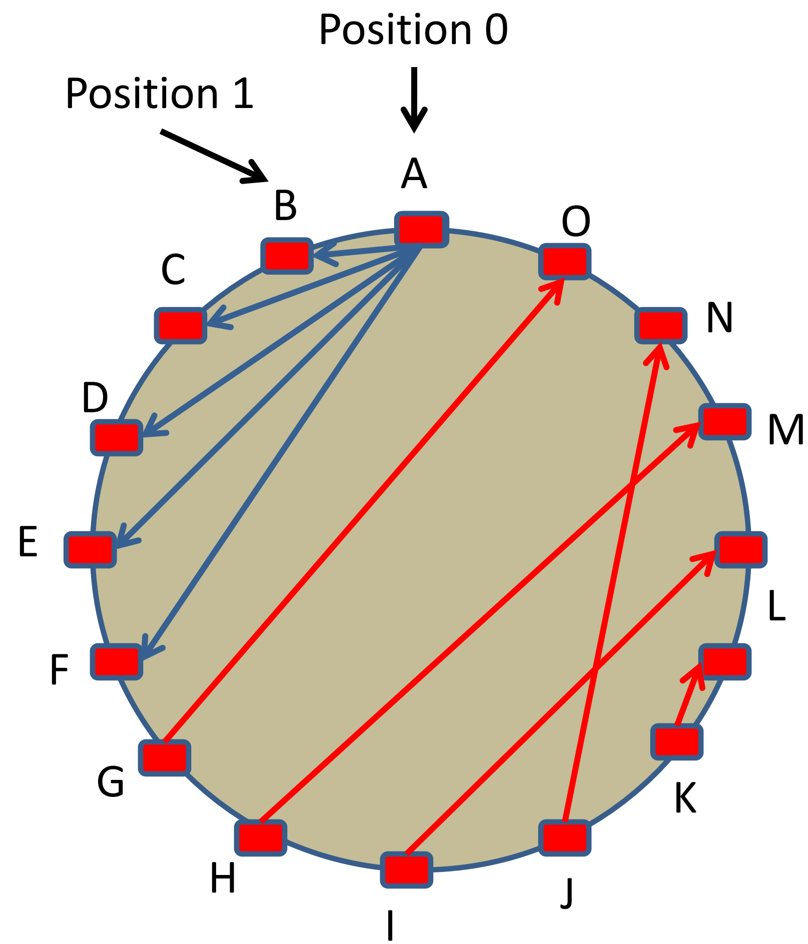
Proof that n/2 + n/4 -1 is optimal for a (very) special case

Nagendra Gulur

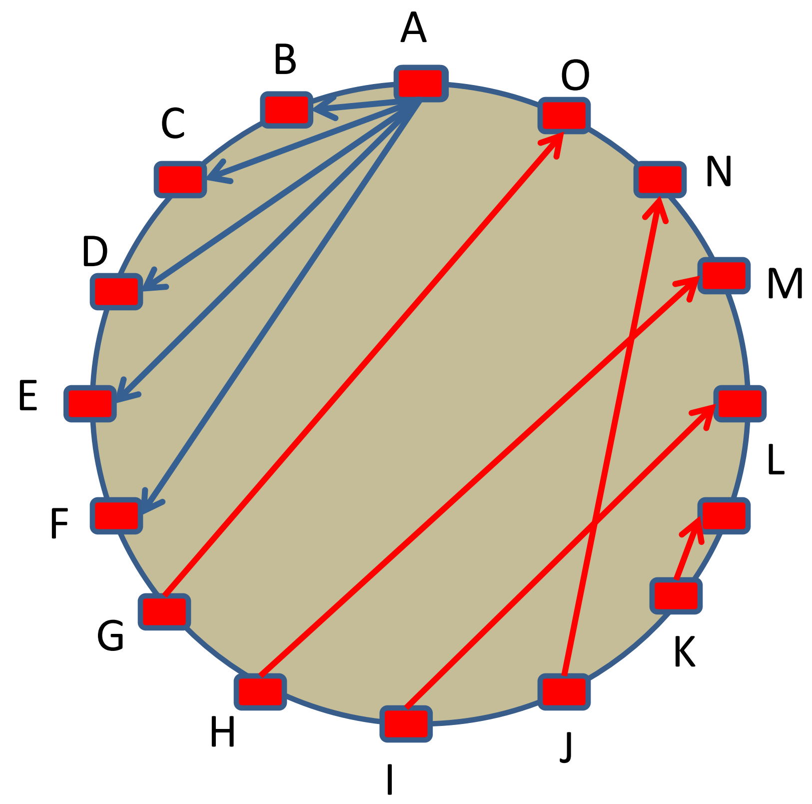
Some Definitions First

- Number the seats starting with 0 for the top center seat and going around the circle anti-clockwise.
- Fix A at position 0. This is an arbitrarily chosen convention.
- We classify into two types:
  – *Fixed*: Hints that reference A
  – *Floating*: Hints that do not reference A
- Fixed hints fully determine positions of the referenced nodes.
  – Eg: if hint is "C is 2 places to right of A" nails C to seat #2.
- Floating hints are less obvious. A full seating assignment may be needed to resolve them.
- See figure on right for n=16, blue hints are fixed, red ones are floating.
- Observe that:
  – B, C, D, E, and F are determined easily.
  – G through O are not determined until all floating hints have been "solved" together.



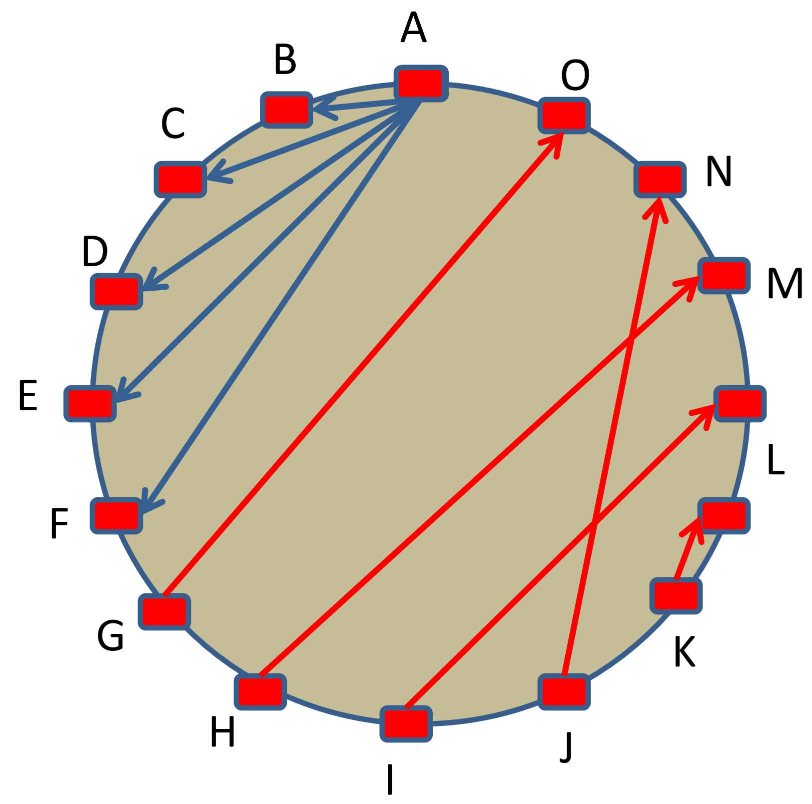
Fixed vs Floating Hints

- We observe that each *fixed* hint fixes one person's seat position.
- Whereas each *floating* hint *can* fix two positions.
  - In the figure, each floating hint pegs two people's positions.
  - (However, note that by our definition floating does not mean it *has to* fix two positions.)



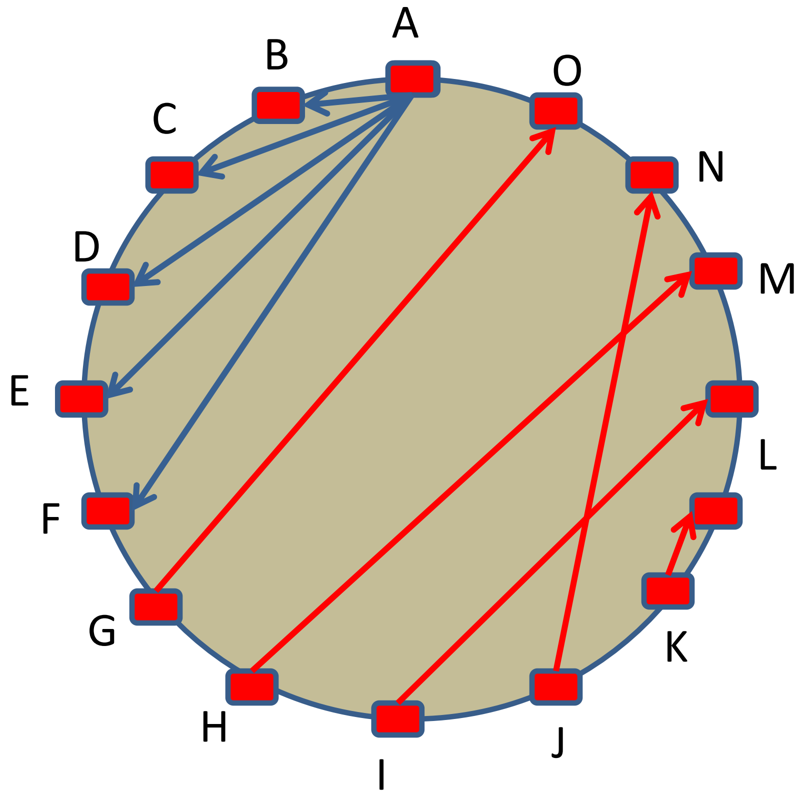
Our Special Case – *Clustered fixed* hints

- *All fixed hints are clustered together adjacent to A clockwise or anti-clockwise.*
- See picture for an example.
- We use this picture as running example of proof.
- Here there are k=5 fixed hints that are clustered.
- *Our claim: If fixed hints are clustered together, then n/2 + n/4 -1 is tight. Can not do better.*
- If our claim is correct, then the hints in the example do not define a unique seating.
  - We need at least n/2+n/4-1 = 11 hints (n=16)
  - We have only 10 hints in here (5 fixed and 5 floating).
- We show by constructing an alternate solution that satisfies all the hints given.



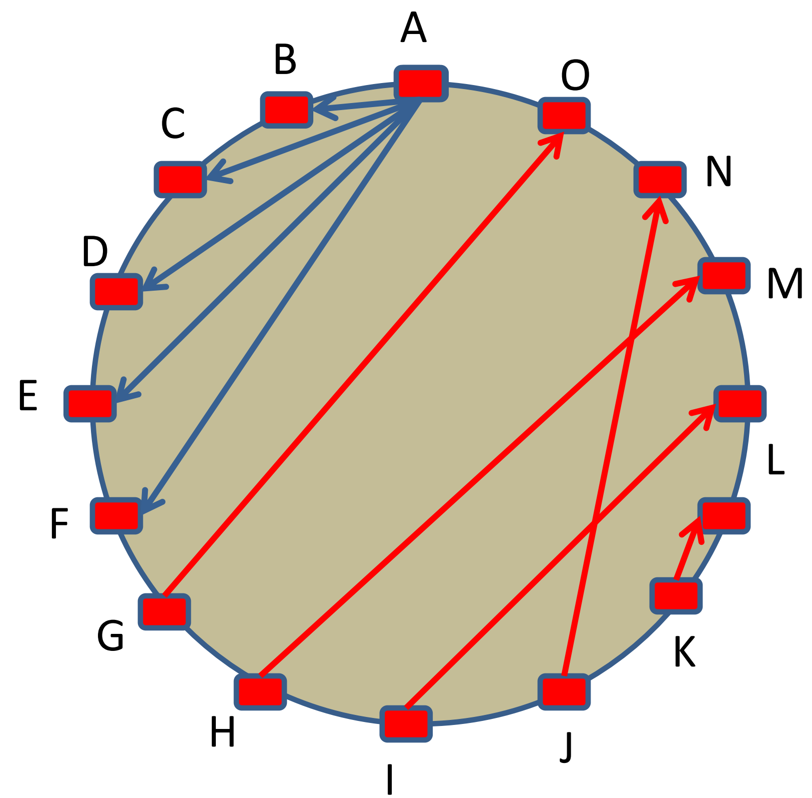
Main Idea

- Our argument is based on constructing an alternate solution from a given solution.
- The only case when there's no alternate is when the floating hints satisfy a certain invariance.
- And when that happens, then we can not have more than n/4 floating.
- And our proof follows after some technicalities are resolved.
- Details follow.

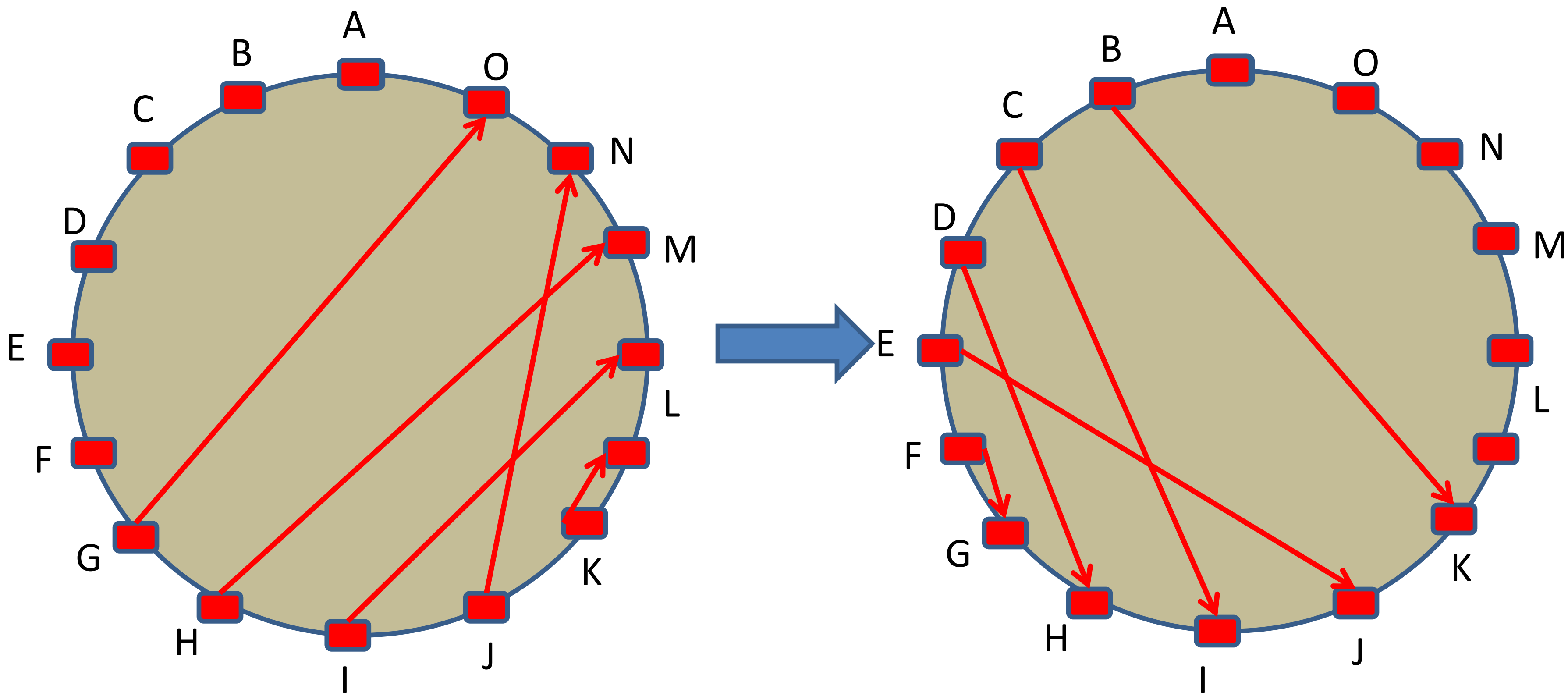


Rotation of floating hints

- General construction.
- Assume k clustered fixed hints.
  - They determine first (k+1) positions ('A' included)
- In example, k=5, so 6 vertices are determined.
- Ignore these fixed hints for now.
- Rotate floating hints by k positions clockwise. See next slide.



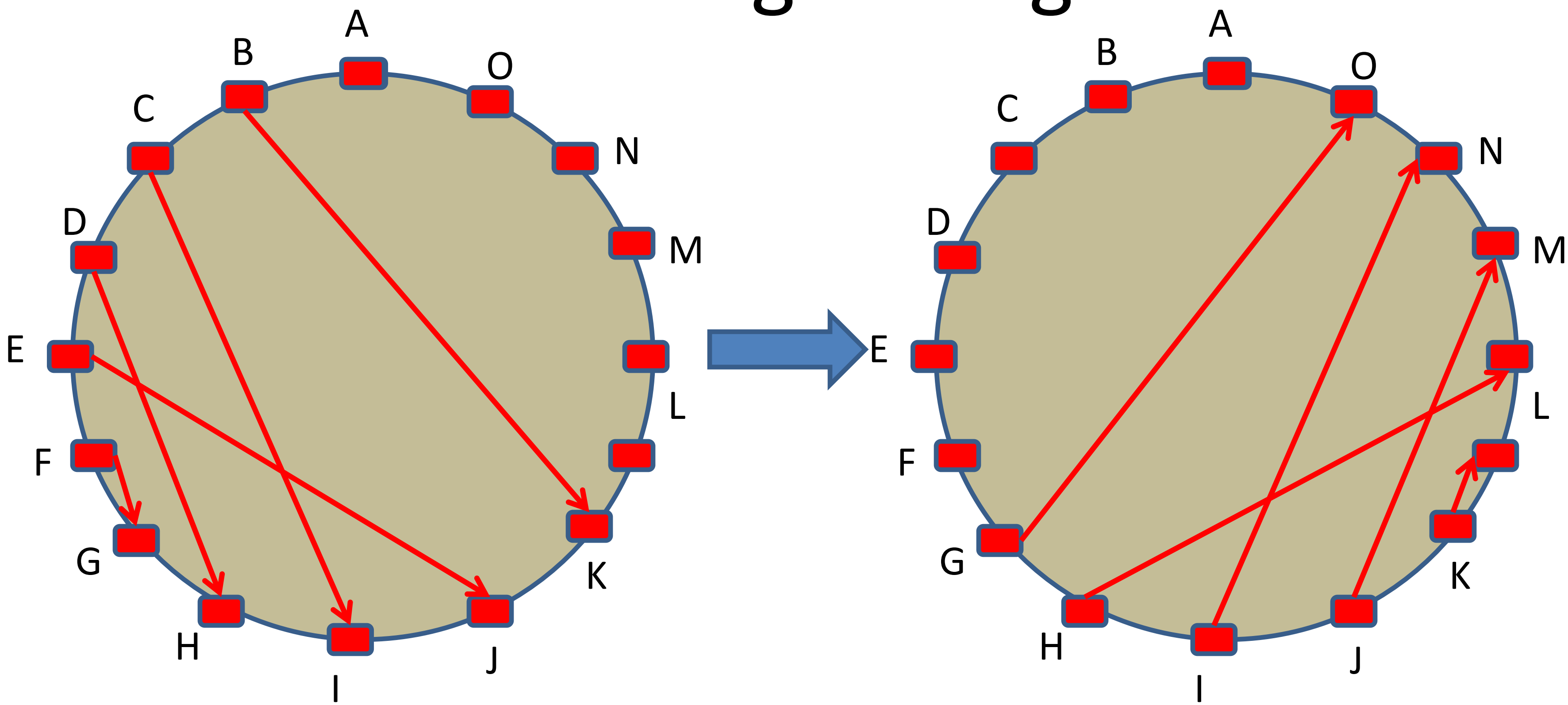
Rotation Example



Before Rotation

After Rotation
(by 5 positions clockwise)

Now Reflect this around the vertical axis running through A



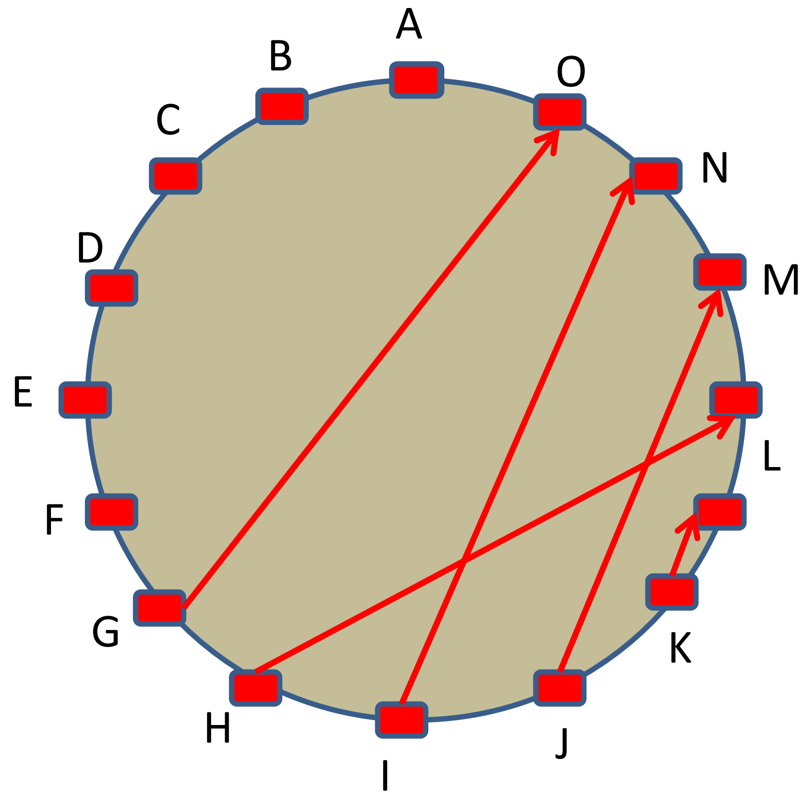
Before Reflection

After Reflection

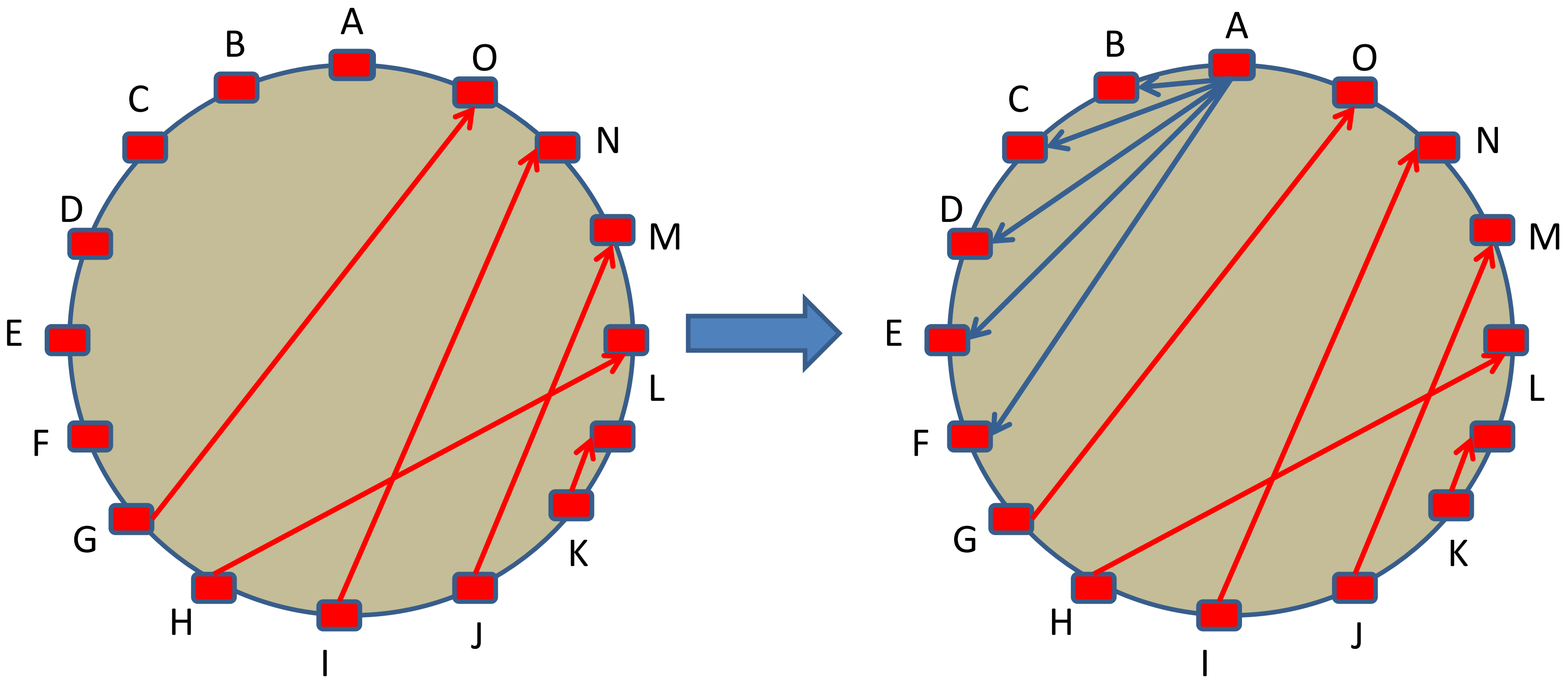
Note: Since reflection changes clockwise to anticlockwise, we restore the original directions by changing the arrow signs

Observe our construction

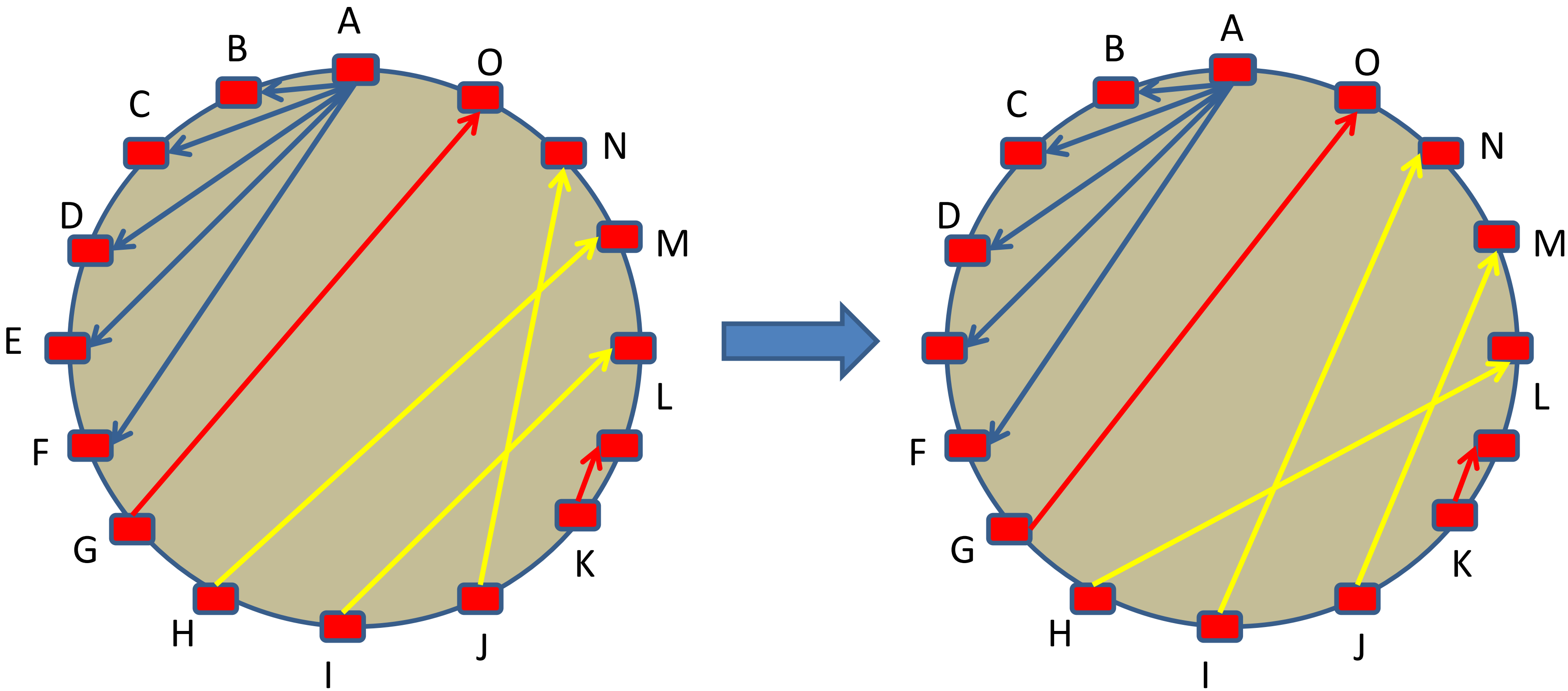
- Since we have only rotated and reflected (all) the floating hints, they preserve the hint distances and relative ordering.
- Thus the new solution we get is valid and satisfies all the hints.
- Our construction does not step on the fixed hint positions.
- Positions A through F in the example are left open.
- We fill in the fixed hints back in.
- See next slide.



Now add the fixed hints back in



Now Look at the Two Solutions!



Original Solution

New Solution

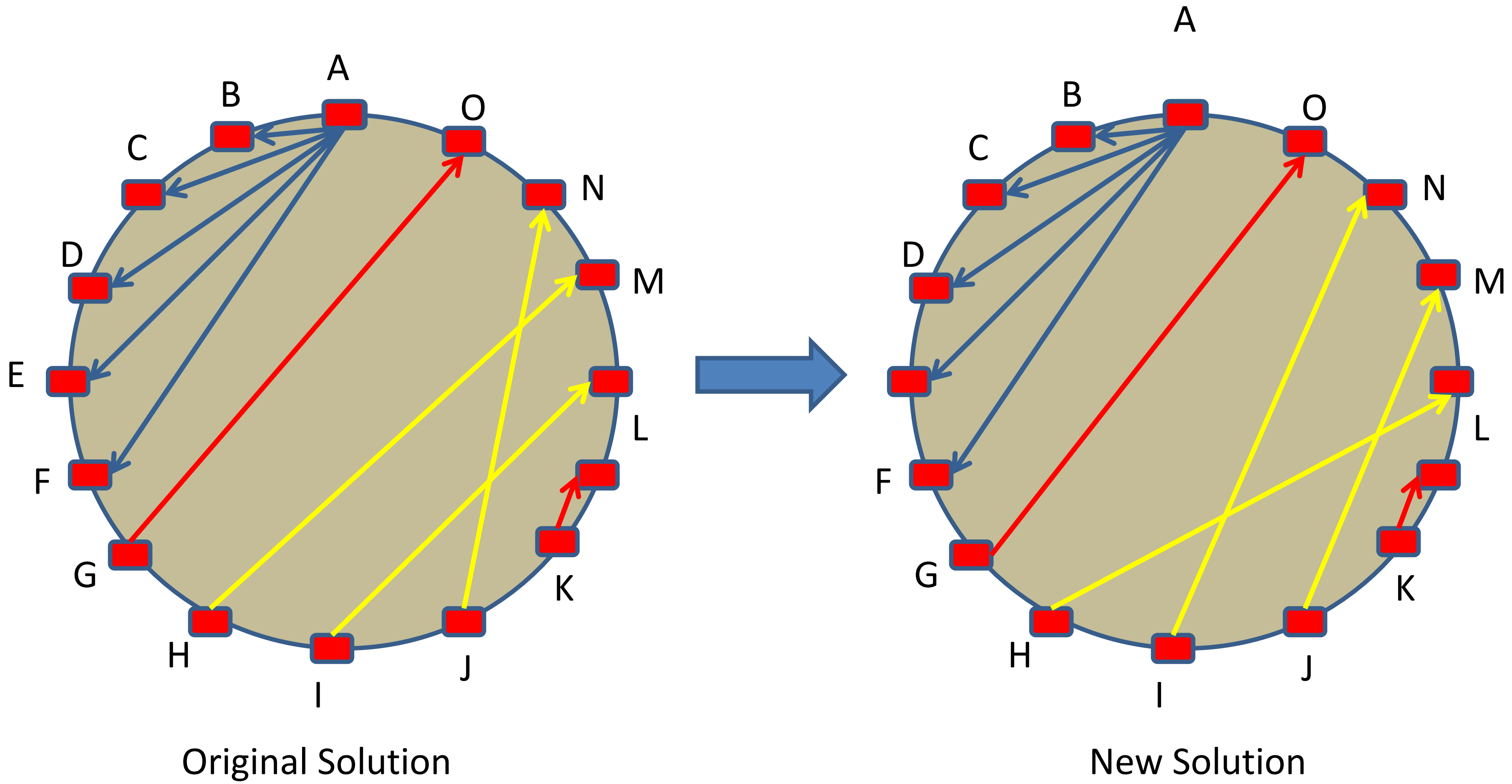
These are different in the highlighted seating assignments shown as yellow hints

Thus we revealed that the original seating assignment was not unique

- Are we done?
- No. We never used the fact that there were only k (where $k < n/2-1$) fixed hints
- To establish the lower bound, we need to examine the condition on a floating hint remaining invariant under rotation+reflection.

Condition for Invariance of Floating Hint

- Look at the two solutions again.

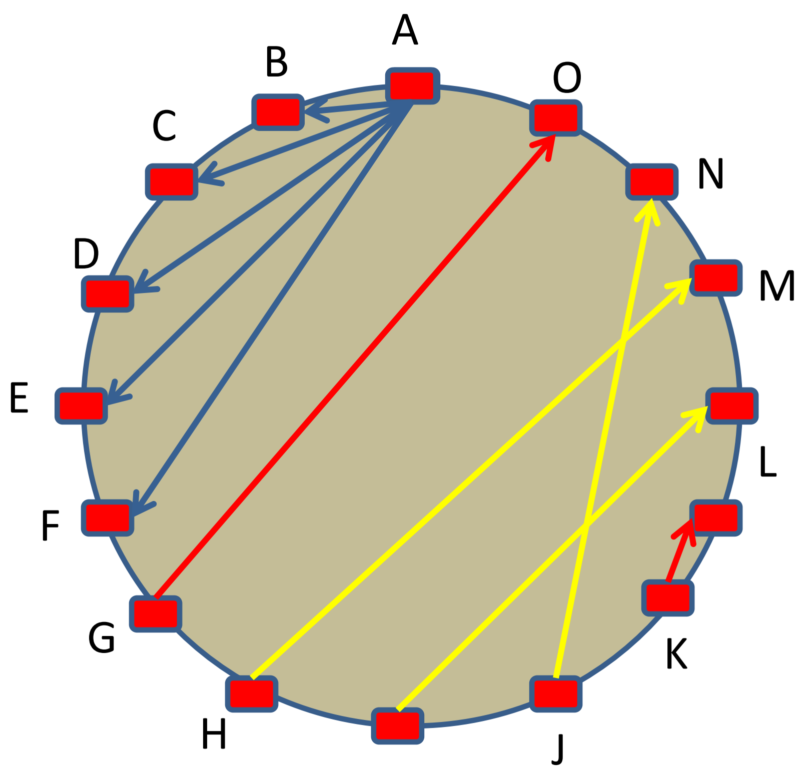


- Why do two of the floating hints (in red) remain in exactly the same positions before and after the transformation?

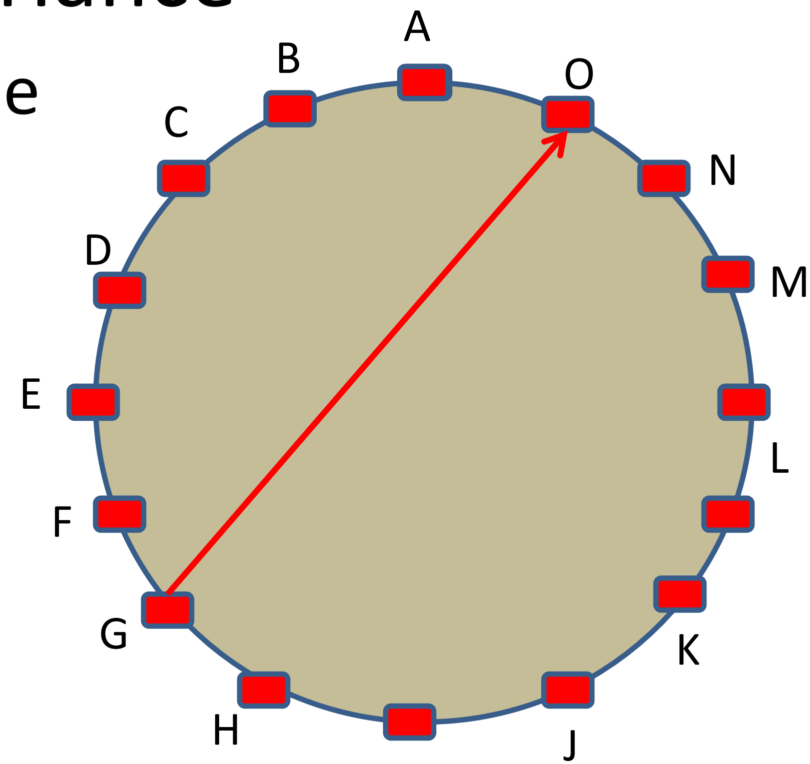
Condition for Invariance of Floating Hint

- A floating hint remains invariant across the transformation (rotation + reflection) whenever the vertices specified are of the form: (k+j, n-j)
  - That is, vertex (n-j) is to right of (k+j) by (n-k-2j) distance.
- In the figure,
  - (G,O) == (6,15) == (5+1,16-1)
  - (K,L) == (10,11) == (5+5,16-5)
- It can be verified that

only such hints remain invariant



Proof of Invariance



- Take (6,15) as working example
- In general, take a (k+j, n-j)
- After rotation by k steps, it

becomes (j,n-j-k)

Eg: (6,15) → (1,10)

- Now upon reflection (j, n-j-k)

becomes (n-j, k+j)

Eg: (1,10) becomes (15,6)

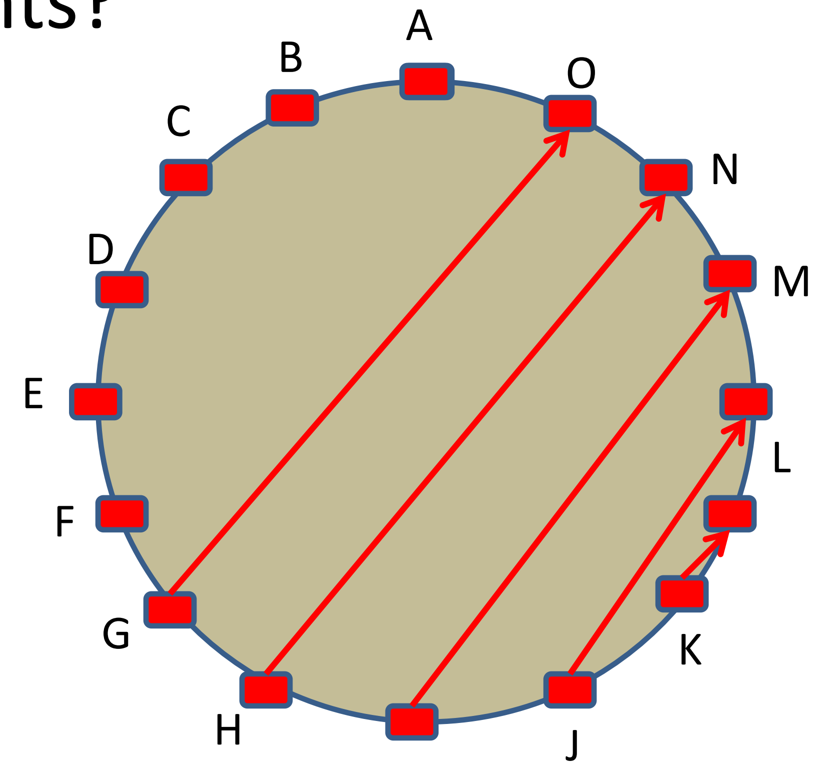
- Since we flip directions, we are back to (6, 15)
- Thus such assignments remain invariant under the transformation.

So What did we establish so far?

- We establish that if a hint is invariant, then it achieves identical seating before and after the transformation.
- Thus – if any hint is not invariant, then we get an alternate solution and thus our original seating is not unique. We are done.
  - This argument does not depend on having $k < n/2-1$ clustered fixed hints. It is always true for whatever k clustered fixed hints.
- So the case to consider is when all hints are invariant.
- Let us look at this case now.

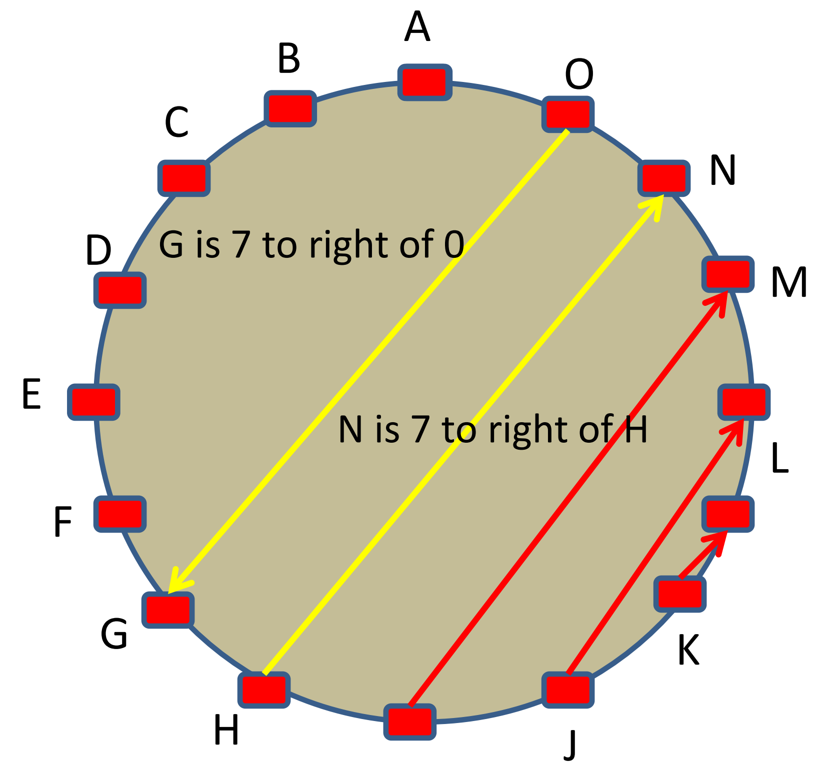
How does a solution look with only invariant floating hints?

- Recall that all invariant hints are of the form (k+j, n-j)
- Hints look like the example on the right.
  - K=5



How many invariant hints are possible?

- Since the invariant hints run "parallel", the distances are either all even or all odd.
- In example, the distances are all odd.
- In general, such distances are either {1,3,5,… (n-1)} or {2,4,6,8,..(n-2)} (assuming n even)
- Thus utmost n/2 hint distances
- However, distance 1 == distance 15, distance 2 == distance 14 etc
- Because going a distance 1 from X to Y is equal to going distance 15 from Y to X.
- We can not distinguish between such pairs.
- Thus utmost n/4 unique invariant hints possible!
- In figure for example, hint (O,G) is identical to hint (H,N) and thus this is not unique.



Let’s close our argument

- For invariance under our transformation, we require only invariant hints.
- We can have utmost n/4 such hints.
- These hints cover n/2 nodes (each hint pegs 2 positions)
- How many fixed hints do we need then to fully specify the other nodes?
  - Recall: Each fixed hint fixes one position.
  - k Hints ➔ k+1 positions, including position 0 for ‘A’
  - If k < n/2-2, then totally we cover utmost n/2-2 + n/2 = n-2 nodes. Two nodes not covered. So solution not unique.
  - If k =n/2-2, then one invariant hint goes from (k+1, n-1) == (n2/-1,n-1) whose distance is n/2 (assume n even)
    - This hint is ambiguous since it is exactly half-way round the circle.
    - Could go from either node to the other and satisfy the hint
    - Thus k=n/2-2 is not enough either
  - Thus we need at least k=n/2-1 hints

Summary of Proof

- Assume Clustered fixed hints
- Defined a transformation of rotation + reflection
- Defined invariance of a hint under transform.
- If any hint is not invariant then transformation gives a new solution. Thus original solution not unique.
- If all hints invariant, then we showed we can have utmost n/4 of them.
  - Then we showed that we need at least n/2-1 fixed hints to fully specify unique seating.
  - Thus n/2+n/4-1 is tight for the special case of clustered fixed hints.