Scanning Cats

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The m cool cats of the Catatonic Choir put on a show at the Catacoustic Centrium. Decorated in catmints and catnips, they emerged from backstage through a catwalk, to a round of enthusiastic applause. It went downhill from there. At first, the audience began to take catnaps. Then catcalls were heard. Their confidence shaken, the members allowed the final chorus to degenerate into a caterwaul. Various categories of objects, some launched from catapults, rained on the stage. It was a catastrophic performance. The local paper, *Cat of Nine Tales*, whipped the choir with a cataract of catacaustic comments.

The conductor came to the conclusion that there were copy cats in the choir, masquerading as cool cats. The only way to tell them apart was to use the catscans from the Catoptrics Consortium. The cheapest model could tell the exact number of copy cats among those scanned, but without identifying them individually. As a promotion, the company offered one free use of a large catscan which could take in any number of cats at a time. A group scan revealed that there were indeed n copy cats among them, where $0 < n < m$. A small catscan was then rented. It took in exactly k cats at a time, $1 < k < m$. Since the fee per use was extremely high, and the choir's budget was smaller than a budgie, it was essential that the catscan should be used as few times as possible.

Denote by $f(m, n, k)$ the minimum number of times the catscan must be used. The special case $f(20, 4, 3)$ was featured in an Omniheurists' Contest when *The Puzzling Adventures of Dr. Ecco* was first published by W. H. Freeman. The best submitted answer, according to author Dennis Shasha of the Courant Institute, was $f(20, 4, 3) \leq 10$, embodied as a 71-page computer printout with no explanations. When the book was republished by Dover later, Dennis asked that the reference to the contest be removed. A over-zealous editor deleted the contest problems as well.

In this paper, we present a readable proof that $f(20, 4, 3) \leq 11$. Number the cats from 1 to 20. Scan the first six of the following groups, namely, (1,2,3), (4,5,6), (7,8,9), (10,11,12), (13,14,15), (16,17,18) and (19,20). There are four cases.

Case 1. The distribution of the copy cats is 3-1.
We may assume that 1, 2 and 3 are copy cats.

Subcase 1(a). The other copy cat is one of 19 and 20.
The seventh scan (1,2,19) yields two or three copy cats. This allows us to determine whether 19 or 20 is the other copy cat. So seven scans suffice here.

Subcase 1(b). The other copy cat is one of 4, 5 and 6.
The seventh scan is (4,19,20) and the eighth scan is (5,19,20). If either yields one copy cat, then it is 4 in the former case and 5 in the latter case. If neither yields any copy cats, then 6 is a copy cat. So eight scans suffice here.

Case 2. The distribution of the copy cats is 2-2.
We may assume that two of 1, 2 and 3 are copy cats.

Subcase 2(a). The other copy cats are 19 and 20.
The seventh scan is (1,19,20). If it yields two copy cats, then 2 and 3 are copy cats. Otherwise, 1 is a copy cat. The eighth scan (2,19,20) also yields two or three copy cats. This allow us to tell whether 2 or 3 is the remaining copy cat. So eight scans suffice here.

Subcase 2(b). The other copy cats are two of 4, 5 and 6.
This is the complement of Subcase 3(a) below.

Case 3. The distribution of the copy cats is 2-1-1.

Subcase 3(a). The copy cats are 19, 20, one of 1, 2 and 3 and one of 4, 5 and 6.
The seventh scan is (1,4,19) and the eighth scan is (2,5,20). Each yields one, two or three copy cats, as summarized in the following chart.

		Seventh		
		1	2	3
E i	1	(3,6) are copy cats	(1,6) or (3,4)	(1,4) are copy cats
g h	2	(2,6) or (3,5)	(1,5) or (2,4)	Not Possible
t h	3	(2,5) are copy cats	Not Possible	Not Possible

A ninth scan, chosen from (1,19,20), (2,19,20) and (3,19,20), will settle the remaining uncertainties.

Subcase 3(b). The copy cats are two of 1, 2 and 3, one of 4, 5 and 6 and one of 19 and 20.
This can be handled as in Subcase 3(c) below, if we pretend that we do not know 18 is a cool cat.

Subcase 3(c). The copy cats are two of 1, 2 and 3, one of 4, 5 and 6 and one of 7, 8 and 9.
The seventh scan is (1,4,7) and the eighth scan is (2,4,7). If they yield the same number of copy cats, then 1 and 2 are copy cats. Otherwise, 3 is a copy cat, and we can also tell which of 1 and 2 is a copy cat. Moreover, we know how many copy cats are among 4 and 7. It may happen that both are copy cats. If neither is a copy cat, then two more scans, namely (5,19,20) and (8,19,20), will identify the remaining copy cats. Suppose exactly one of 4 and 7 is a copy cat. The ninth scan is (4,6,8) and the tenth scan is (5,7,8). Each yields zero, one or two copy cats, as summarized in the following chart.

		Ninth		
		0	1	2
T e	0	Not Possible	(4,9) are copy cats	Not Possible
n t	1	Not Possible	(6,7) are copy cats	(4,8) are copy cats
h	2	(5,7) are copy cats	Not Possible	Not Possible

Case 4. The distribution of the copy cats is 1-1-1-1.
We may assume that one of 1, 2 and 3, one of 4, 5 and 6 and one of 7, 8 and 9 are copy cats.

Subcase 4(a). The other copy cat is one of 19 and 20.
This can be handled as in Subcase 4(b) below, as though we do not know that 18 is a cool cat. Interchange the labels (10,11,12) and (18,19,20).

Subcase 4(b). The other copy cat is one of 10, 11 and 12.
The seventh scan is (1,7,10), the eighth scan is (2,7,10), the ninth scan is (4,8,11) and the tenth scan is (5,8,11). As in Subcase 3(c), we know which of 1, 2 and 3 is a copy cat, and which of 4, 5 and 6 is a copy cat. Moreover, we know the number of copy cats among 7 and 10, and among 8 and 11. Each is zero, one or two, as summarized in the following chart.

		7 and 10		
		0	1	2
8 and 11	0	(9,12) are copy cats	(7,12) or (9,10)	(7,10) are copy cats
	1	(8,12) or (9,11)	(7,11) or (8,10)	Not Possible
	2	(8,11) are copy cats	Not Possible	Not Possible

An eleventh scan, chosen from (7,19,20), (8,19,20) and (9,19,20), will settle the remaining uncertainties.