Efficient Scan

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1 Introduction

An interest problem of *cat scan* was featured in an Omniheurists' Contest when *The puzzling Adventures of Dr. Eco* was first published by W. H. Freeman.

Assume there are m cats, among which there are n copy cats and $m-n$ cool cats, where $0 < n < m$. A small catscan can take exactly k cats at a time, $1 < k < m$, and tell the exact number of copy cats among these k cats, but without identifying them individually.

Denote by $f(m, n, k)$ the minimum number of times the catscan must be used to tell all n copy cats. Yen-Kang Fu from Taipei proves that $f(10, 4, 3) \leq 11$. In this article, we would show that $f(10, 4, 3) \leq 10$.

2 Assumption

In the original problem, the catscan takes exactly $k = 3$ cats at a time. In this article, we assume we have identified at least 2 real cool cats already, which means that we can choose 1, 2, or 3 cats for a scan.

During our analysis, this assumption is always satisfied whenever we choose less than 3 cats for a scan.

3 Notation

We denote by $\{(a_1, b_1), (a_2, b_2), \ldots, (a_s, b_s)\}$ a group of cats that contains s disjoint subgroups, where the i-th subgroup contains b_i copy cats and $a_i - b_i$ cool cats, thus there are $\sum_{i=1}^{s} a_i$ cats in the whole group.

Also, we assign $x_i = 1$ if the i-th cat is a copy cat and $x_i = 0$ if it is a real cool cat. So the sum of x_i's is the exact number of copy cats in a group.

4 Tactics

In this section we discuss some useful tactics during the scan.

Lemma 1. *If x_1 is unknown, and $x_2 + x_3 + x_4$ is given, then all four cats can be identified after two scan. We describe that x_1 forms a* one-cat detector*.*

Proof. The cases for $x_2 + x_3 + x_4 = 0$ or 3 are trivial.

For $x_2 + x_3 + x_4 = 1$ or 2, the two scans are (x_1, x_2) and (x_1, x_3). If both yield the same number, then $x_2 = x_3$ so we can decide all values of x_2, x_3, and x_4. If both scans yield different numbers, then we know which one, x_2 or x_3, is 1 and another is 0, as well as x_4 since their sum is given. After x_2 and x_3 are identified, the value of x_1 can be determined as well. □

Lemma 2. *If $x_1 + x_2 = 1$, $x_2 + x_3 = 1$, and x_4 is unknown, then all four cats can be identified after one scan. We describe that x_1 and x_3 form a* two-cat detector*.*

Proof. Obviously $x_1 = x_3$ since both equal to $1 - x_2$. We scan (x_1, x_3, x_4), then the result would be the binary number $\overline{x_1 x_4}$. That is, $x_1 = x_3 = 0$ and $x_4 = 1$ yields 1, $x_1 = x_3 = 1$ and $x_4 = 0$ yields 2, $x_1 = x_3 = x_4 = 0$ yields 0, and $x_1 = x_3 = x_4 = 1$ yields 3. Thus x_1, x_3, and x_4 can be identified in one scan, as well as x_2 which equals to $1 - x_1$. □

Lemma 3. *If $x_1 + x_2 = 1$, $x_2 + x_3 = 1$, $x_3 + x_4 = 1$, and x_5 is unknown, then all five cats can be identified after one scan. We describe that x_1, x_2, x_3, and x_4 form a* four-cat detector*.*

Proof. Since x_1 and x_3 form a two-cat detector, so x_1, x_3, and x_5 can be identified in one scan, as well as x_2 and x_4 which equal to x_1. □

Lemma 4. *If $x_1 + x_2 + x_3 = 1$, $x_4 + x_5 + x_6 = 1$, and x_7 is unknown, then all seven cats can be identified after three scans. We describe that x_1, x_2, x_3, x_4, x_5, and x_6 form a* six-cat detector*.*

Proof. The first two scans are (x_1, x_4) and (x_2, x_5). If either scan yields 2, then we identify two copy cats and the third scan on x_7 alone suffices.

If both scans yields 0, then x_3 and x_6 are copy cats and the third scan on x_7 alone suffices.

If one scan yields 1 and another yields 0, then $x_3 + x_6 = 1$; if both scans yield 1, then $x_3 + x_6 = 0$. In both cases, we have found out a four-cat detector from the first six cats, so one more scan suffices. □

Lemma 5. *If x_1 and x_2 are unknown, and $x_3 + x_4 + x_5$ is given, then all values of x_3, x_4, x_5, and $x_1 + x_2$ can be determined after two scan. We describe that x_1 and x_2 form a* two-cat estimator*.*

Proof. This is a special case of one-cat detector if we let $y = x_1 + x_2$. □

Lemma 6. *If $x_1 + x_2 + x_3 = 1$ and $x_3 + x_4 + x_5 = 2$, then all values of x_3, x_4, x_5, and $x_1 + x_2$ can be determined after one scan. We describe that these five cats form* V-1-2*.*

Proof. The scan is (x_1, x_2, x_4). Using the same argument for one-cat detector, we can determine all values of x_3, x_4, x_5, and $x_1 + x_2$ after the scan. $\square$

Lemma 7. *If $x_1 + x_2 + x_3 = 1$, $x_3 + x_4 + x_5 = 1$, and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 3$, then all values of x_3, $x_1 + x_2$, $x_4 + x_5$, $x_6 + x_7$, and $x_8 + x_9$ can be determined after two scans.*

Proof. The first scan is (x_3, x_6, x_7).

1. If $x_3 + x_6 + x_7 = 3$

   This means that $x_3 = x_6 = x_7 = 1$ and other variables are 0.

2. If $x_3 + x_6 + x_7 = 0$

   This implies that $x_3 = x_6 = x_7 = 0$ and $x_1 + x_2 = x_4 + x_5 = 1$. One more scan on (x_8, x_9) yields the value of $x_8 + x_9$.

3. If $x_3 + x_6 + x_7 = 2$

   If $x_3 = 0$, then we would have at least 4 copy cats since $x_1 + x_2 = x_4 + x_5 = x_6 = x_7 = 1$, which is impossible as their sum is less than four, so we must have $x_3 = 1$ and $x_6 + x_7 = 1$, which implies $x_1 + x_2 = x_4 + x_5 = 0$. One more scan on (x_8, x_9) yields the value of $x_8 + x_9$.

4. If $x_3 + x_6 + x_7 = 1$

   The second scan is (x_3, x_8, x_9).

   (a) If $x_3 + x_8 + x_9 = 3$
   This means that $x_3 = x_8 = x_9 = 1$ and other variables are 0.

   (b) If $x_3 + x_8 + x_9 = 0$
   Here $x_3 = x_8 = x_9 = 0$, so $x_1 + x_2 = x_4 + x_5 = x_6 + x_7 = 1$.

   (c) If $x_3 + x_8 + x_9 = 2$
   Using the same argument in the last scan, we must have $x_3 = 1$, and as a result $x_8 + x_9 = 1$ and $x_1 + x_2 = x_4 + x_5 = x_6 + x_7 = 0$.

   (d) If $x_3 + x_8 + x_9 = 1$
   We must have $x_3 = 1$, otherwise we have 4 copy cats. As a result, all other eight variables are 0.

$\square$

Definition 1. *If $x_1 + x_2 + x_3 = 1$ and $x_3 + x_4 + x_5 = 1$, then we describe that these five cats form* V-1-1.

5 Endgames

Some endgames are encountered many times later so we analyse them here to avoid repeated work.

Let S be a set of cats. Denote by $g(S)$ the minimum number of scans to identify all copy cats in S.

5.1 $S = \{(m,1)\}$

Obviously $g(\{(1,1)\}) = 0$, $g(\{(2,1)\}) = 1$, $g(\{(3,1)\}) = 2$, $g(\{(4,1)\}) = 2$, $g(\{(5,1)\}) = 3$.

For $m \geq 6$, $g(\{(m,1)\}) = \max(g(\{(m-3,1)\}), g(\{(3,1)\})) + 1 = g(\{(m-3,1)\}) + 1$. The reason is, we scan 3 cats from these m cats. If it yields 1, then we need $g(\{(3,1)\})$ more scans; otherwise we need $g(\{(m-3,1)\})$ more scans. However, $g(\{(m-3,1)\}) \geq g(\{(3,1)\})$ for $m \geq 6$, since $m-3 \geq 3$ and $g(\{(x,1)\})$ is a non-decreasing function.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g(\{(m,1)\})$	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7

5.2 $S = \{(3,1),(2,1)\}$ or $\{(3,2),(2,1)\}$

Lemma 8. *If $x_1 + x_2 = 1$ and $x_3 + x_4 + x_5$ is given, then all five cats can be identified after two scans.*

Proof. The first scan is (x_2, x_3). If it yields 0 or 2, then x_1, x_2, and x_3 are identified, so one more scan on x_4 will suffice.

If the first scan yields 1, then x_1 and x_3 is a two-cat detector, so one more scan will identify the first four cats, as well as x_5 since $x_3 + x_4 + x_5$ is given. $\square$

5.3 $S = \{(3,2),(3,2)\}$

This is the complement case of $\{(3,1),(3,1)\}$, which is a six-cat detector, so 3 scans suffice.

5.4 $S = \{(2,1),(m,1)\}$

The cases for $m = 1,2,3$ have been discussed, and $g(\{(2,1),(4,1)\}) \leq g(\{(2,1),(2,1)\}) + 1 = 3$, $g(\{(2,1),(5,1)\}) \leq g(\{(2,1),(3,1)\}) + 1 = 3$.

For $m \geq 6$, with similar argument for $S = \{(m,1)\}$, we have $g(\{(2,1),(m,1)\}) \leq g(\{(2,1),(m-3,1)\}) + 1$.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(\{(2,1),(m,1)\})$	2	2	3	3	3	4	4	4	5	5	5	6	6	6

5.5 $S = \{(3,1),(m,1)\}$

The cases for $m = 1,2,3$ have been discussed, and $g(\{(3,1),(4,1)\}) \leq g(\{(3,1),(2,1)\}) + 1 = 3$, $g(\{(3,1),(5,1)\}) \leq g(\{(3,1),(3,1)\}) + 1 = 4$.

For $m \geq 6$, with similar argument for $S = \{(m,1)\}$, we have $g(\{(3,1),(m,1)\}) \leq g(\{(3,1),(m-3,1)\}) + 1$.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(\{(3,1),(m,1)\})$	2	3	3	4	4	4	5	5	5	6	6	6	7	7

5.6 $S = \{(m, 2)\}$

Obviously $g(\{(2,2)\}) = 0$, $g(\{(3,2)\}) = 2$, $g(\{(4,2)\}) = 3$, $g(\{(5,2)\}) = 3$.

For $m \geq 6$, we $g(\{(m,2)\}) \leq \max(g(\{(m-3,2)\}), g(\{(3,1),(m-3,1)\}), g(\{(3,2)\}))+1$, with similar arguments in $S = \{(m,1)\}$.

m	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g(\{(m,2)\})$	2	3	3	4	4	5	5	5	6	6	6	7	7	7	8

5.7 $S = \{(2,1),(2,1),(2,1)\}$

Lemma 9. *If $x_1 + x_2 = 1$, $x_3 + x_4 = 1$, and $x_5 + x_6 = 1$, then all six cats can be identified after two scans.*

Proof. The first scan is (x_2, x_4). If it yields 0 or 2, then the first four cats have been identified and one more scan on x_5 suffice.

If $x_2 + x_4 = 1$, then the first four cats form a four-cat detector, so one more scan can identify the first fives cats, as well as x_6 which equals to $1 - x_5$. □

5.8 $S = \{(3,1),(3,1),(2,1)\}$ or $\{(3,2),(3,1),(2,1)\}$

Lemma 10. *If $x_1 + x_2 + x_3$ is given, $x_4 + x_5 + x_6 = 1$, and $x_7 + x_8 = 1$, then all eight cats can be identified after three scans.*

Proof. We use x_6 and x_8 as a two-cat estimator. After two scans, we know all values of the first three cats as well as the value of $x_6 + x_8$. Using similar argument for $S = \{(3,1),(2,1)\}$, we know that one more scan suffice. □

5.9 $S = \{(3,1),(3,1),(4,1)\}$ or $\{(3,2),(3,1),(4,1)\}$

In this case, four scans suffice.

The reason is, after one scan on two cats in the 4-cat subgroup, the endgame becomes $\{(3,n),(3,1),(2,1)\}$, so three more scans suffice.

5.10 $S = \{(3,1),(3,1),(7,1)\}$ or $\{(3,2),(3,1),(7,1)\}$

In this case, five scans suffice.

With the same argument in the previous endgame, after one scan on three cats in the 7-cat subgroup, the endgame becomes $\{(3,n),(3,1),(3,1)\}$ or $\{(3,n),(3,1),(4,1)\}$, so four more scans suffice.

5.11 $S = \{(3,1),(3,1),(3,1)\}$ or $\{(3,1),(3,1),(3,2)\}$

Lemma 11. *If $x_1 + x_2 + x_3 = 1$, $x_4 + x_5 + x_6 = 1$, and $x_7 + x_8 + x_9$ is given, then all nine cats can be identified after four scans.*

Proof. The first six cats form a six-cat detector, so the first seven cats can be identified after three scans, then one more scan on x_8 will suffice. $\square$

5.12 $S = \{(3,1),(2,1),(2,1),(2,1)\}$

Lemma 12. *If $x_1 + x_2 + x_3$ is given, $x_4 + x_5 = 1$, $x_6 + x_7 = 1$, and $x_8 + x_9 = 1$, then all nine cats can be identified after three scans.*

Proof. We use x_5 and x_7 as a two-cat estimator. After two scans, we identify all values of the first three cats, as well as the value of $x_5 + x_7$. Using similar arguments for $S = \{(2,1),(2,1),(2,1)\}$, we know one more scan suffice. $\square$

5.13 $S = \{(3,1),(3,1),(3,1),(2,1)\}$

Lemma 13. *If $x_1 + x_2 + x_3 = 1$, $x_4 + x_5 + x_6 = 1$, $x_7 + x_8 + x_9 = 1$, and $x_{10} + x_{11} = 1$, then all eleven cats can be identified after four scans.*

Proof. We use x_3 and x_6 as two-cat estimator, so after two scans we know all values of x_7, x_8, x_9, and $x_3 + x_6$.

Now the first six cat form a six-cat detector. Using the same argument in *six-cat detector*, we could identify all cat after two more scans. $\square$

5.14 $S = \{(11,4)\}$

Lemma 14. *If $\sum_{i=1}^{14} = 4$, then all eleven cats can be identified after seven scans.*

Proof. We separate all 11 cats into four subgroups (x_1, x_2), (x_3, x_4, x_5), (x_6, x_7, x_8), and (x_9, x_{10}, x_{11}). The first three scans are the first three subgroups, so that we know the distribution of 4 copy cats in these four subgroups. The distribution may be 3-1, 2-2, 2-1-1, or 1-1-1-1, but all these endgames have been analysed and at most 4 more scans suffice. $\square$

5.15 $S = \{$V-1-1$, (3,n)\}$, where n is unknown

Lemma 15. *If $x_1 + x_2 + x_3 = x_3 + x_4 + x_5 = 1$ and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 4$, then all eight cats can be identified after three scans.*

Proof. Since $x_1 + x_2 + x_3 + x_4 + x_5 \leq (x_1 + x_2 + x_3) + (x_3 + x_4 + x_5) = 2$, so $x_6 + x_7 + x_8 \geq 2$.

Depending on the value of x_3, there are two possibilities: either $x_3 = 1$, then $x_1 = x_2 = x_4 = x_5 = 0$ and the last three cats are copy cats; or $x_3 = 0$, then $x_1 + x_2 = x_4 + x_5 = 1$ and $x_6 + x_7 + x_8 = 2$.

The first two scans are (x_1, x_4, x_6) and (x_1, x_4, x_7). If both yield the same number, then $x_6 = x_7 = 1$; if they yield different numbers, then we know which one, x_6 or x_7, is 1 and the other is 0. What is more, we know the value of $x_1 + x_4$ after these two scans.

There are three cases:

1. If $x_1 + x_4 = 2$, then $x_2 = x_3 = x_5 = 0$, so we identify the first seven cats, as well as x_8 as their sum is 4.
2. If $x_1 + x_4 = 1$, then $x_3 = 0$, $x_2 + x_5 = 1$, and $x_6 + x_7 + x_8 = 2$, so x_8 can be valued as $2 - x_6 - x_7$. Now x_1, x_2, x_4, and x_5 form a four-cat detector, so one more scan suffice.
3. If $x_1 + x_4 = 0$, then $x_2 + x_3 = x_3 + x_5 = 1$, so x_2 and x_5 form a two-cat detector, thus one more scan on (x_2, x_5, x_8) would identify all cats.

$\square$

5.16 $S = \{\textbf{V-1-1}, (3, 1), (3, n)\}$, where n is unknown

Lemma 16. *If $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 1$, and $\sum_{i=1}^{11} x_i = 4$, then all eleven cats can be identified after four scans.*

Proof. We use x_1 and x_4 as a two-cat estimator, so after two scans on (x_1, x_4, x_6) and (x_1, x_4, x_7), we know all values of x_6, x_7, x_8, and $x_1 + x_4$.

There are three cases:

1. If $x_1 + x_4 = 2$, then x_1 and x_4 are another two copy cats. The last copy cat is among (x_9, x_{10}, x_{11}), so two more scans suffice.
2. If $x_1 + x_4 = 1$, then $x_3 = 0$ and $x_1 + x_2 = x_4 + x_5 = 1$. Now x_1, x_2, x_4, and x_5 form a four-cat detector, so the third scan scan on (x_2, x_5, x_9) and the fourth scan on x_{10} suffice.
3. If $x_1 + x_4 = 0$, then $x_2 + x_3 = x_3 + x_5 = 1$, so x_2 and x_5 form a two-cat detector, thus the third scan scan on (x_2, x_5, x_9) and the fourth scan on x_{10} suffice.

$\square$

5.17 $S = \{\textbf{V-1-1}, (3, 2), (3, n)\}$, where n is unknown

Lemma 17. *If $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 2$, and $\sum_{i=1}^{11} x_i = 4$, then all eleven cats can be identified after four scans.*

Proof. We use x_1 and x_4 as a two-cat estimator, so after two scans on (x_1, x_4, x_6) and (x_1, x_4, x_7), we know all values of x_6, x_7, x_8, and $x_1 + x_4$.

There are three cases:

1. If $x_1 + x_4 = 2$, then x_1 and x_4 are the other two copy cats.
2. If $x_1 + x_4 = 1$, then $x_3 = 0$ and $x_1 + x_2 = x_4 + x_5 = 1$. Now x_1, x_2, x_4, and x_5 form a four-cat detector, so one more scan suffice.
3. If $x_1 + x_4 = 0$, then $x_2 + x_3 = x_3 + x_5 = 1$, so x_2 and x_5 form a two-cat detector, thus the third scan scan on (x_2, x_5, x_9) and the fourth scan on x_{10} suffice.

□

5.18 $S = \{\textbf{V-1-1}, (6, n)\}$, where n is unknown

Lemma 18. *If $x_1 + x_2 + x_3 = x_3 + x_4 + x_5 = 1$ and $\sum_{i=1}^{11} x_i = 4$, then all eleven cats can be identified after five scans.*

Proof. The first scan is (x_6, x_7, x_8), after which the endgame becomes $\{$V-1-1, $(3, 1), (3, n-1)\}$, $\{$V-1-1, $(3, 2), (3, n-2)\}$, or $\{$V-1-1, $(3, n)\}$, so at more four more scans suffice. □

5.19 $S = \{\textbf{V-1-1}, (3, 1), (6, n)\}$, where n is unknown

Lemma 19. *If $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 1$, and $\sum_{i=1}^{14} x_i = 4$, then all fourteen cats can be identified after five scans.*

Proof. The first five cats form V-1-1, so we know all values of x_3, x_1+x_2, x_4+x_5, $x_9 + x_{10}$, and $x_{11} + x_{12}$ after two scans, as their sum is not more than 3. As a result, we also know the value of $x_{13} + x_{14}$ since the sum is 4.

1. If $x_3 = 0$

   In this case $x_1+x_2 = x_4+x_5 = 1$, so the endgame is $\{(3, 1), (2, 1), (2, 1), (2, 1)\}$, three more scans suffice.
2. If $x_3 = 1$

   Here the endgame becomes $\{(3, 1), (2, 2), (1, 1)\}$ or $\{(3, 1), (2, 1), (2, 1), (1, 1)\}$, so at most three more scans suffice.

□

5.20 $S = \{\textbf{V-1-1}, (3,2), (6,n)\}$, where n is unknown

Lemma 20. *If* $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 2$, *and* $\sum_{i=1}^{14} x_i = 4$, *then all fourteen cats can be identified after four scans.*

Proof. The first scan is (x_3, x_9, x_{10}).

1. If $x_3 + x_9 + x_{10} = 3$

   This is impossible since $x_9 + x_{10} \leq 1$.

2. If $x_3 + x_9 + x_{10} = 0$

   Now the endgame is $\{(2,1), (2,1), (3,2)\}$, so three more scans suffice.

3. If $x_3 + x_9 + x_{10} = 2$

   In this case $x_3 = x_9 + x_{10} = 1$ since $x_9 + x_{10} \leq 1$, and the endgame becomes $\{(2,1), (3,2)\}$, so two more scans suffice.

4. If $x_3 + x_9 + x_{10} = 1$

   The second scan is (x_3, x_{11}, x_{12}).

   (a) If $x_3 + x_{11} + x_{12} = 3$
   This is impossible since $x_{11} + x_{12} \leq 1$.

   (b) If $x_3 + x_{11} + x_{12} = 0$
   This is also impossible as there are too many copy cats, for $x_1 + x_2 = x_3 + x_4 = x_9 + x_{10} = 1$ and $x_6 + x_7 + x_8 = 2$.

   (c) If $x_3 + x_{11} + x_{12} = 2$
   This means that $x_{11} + x_{12} = 1$ and $x_9 + x_{10} = 0$, which implies $x_3 = 1$. So the endgame is $\{(3,2), (2,1)\}$ and two more scans suffice.

   (d) If $x_3 + x_{11} + x_{12} = 1$
   We must have $x_3 = 1$, otherwise we have at least 6 copy cats. As a result, $x_9 + x_{10} = x_{11} + x_{12} = 0$ and $x_{13} + x_{14} = 1$. So the endgame becomes $\{(3,2), (2,1)\}$ and two more scans suffice.

□

5.21 $S = \{\textbf{V-1-1}, (9,n)\}$, where n is unknown

Lemma 21. *If* $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$ *and* $\sum_{i=1}^{14} x_i = 4$, *then all fourteen cats can be identified after six scans.*

Proof. The first scan is (x_6, x_7, x_8), after which the endgame becomes {V-1-1, $(3,1), (6, n-1)\}$, {V-1-1, $(3,2), (6, n-2)\}$, or {V-1-1, $(6,n)\}$, so at more five more scans suffice. □

5.22 $S = \{\textbf{V-1-1}, (3,1), (9,n)\}$, where n is unknown

Lemma 22. *If* $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 1$, *and* $\sum_{i=1}^{17} x_i = 4$, *then all seventeen cats can be identified after six scans.*

Proof. The first scan is (x_9, x_{10}, x_{11}).

1. If $x_9 + x_{10} + x_{11} = 3$

   This means that there are at least 5 copy cats, so it is impossible.

2. If $x_9 + x_{10} + x_{11} = 0$

   The endgame becomes $\{\text{V-1-1}, (3,1), (6,n)\}$, thus five more scans suffice.

3. If $x_9 + x_{10} + x_{11} = 2$

   Now $x_3 = 0$ implies there are at least 5 copy cats, so $x_3 = 1$ and the endgame is $\{(3,1), (3,2)\}$, three more scans suffice.

4. If $x_9 + x_{10} + x_{11} = 1$

   The second scan is (x_3, x_{12}, x_{13}).

   (a) If $x_3 + x_{12} + x_{13} = 3$
   This implies there are at least 5 copy cats, so it is impossible.

   (b) If $x_3 + x_{12} + x_{13} = 0$
   Here $x_3 = 0$ and $x_1 + x_2 = x_4 + x_5 = 1$, so the endgame becomes $\{(3,1), (3,1), (2,1), (2,1)\}$, four more scans suffice.

   (c) If $x_3 + x_{12} + x_{13} = 2$
   If $x_3 = 0$, there are at least 6 copy cats, so it must be $x_3 = 1$ and $x_{12}+x_{13} = 1$. The endgame is $\{(3,1), (3,1), (2,1)\}$, three more scans suffice.

   (d) If $x_3 + x_{12} + x_{13} = 1$
   Again $x_3 = 0$ implies too many copy cats, so $x_3 = 1$ and $x_{12} = x_{13} = 0$. The endgame becomes $\{(3,1), (3,1), (4,1)\}$, four more scans suffice.

□

5.23 $S = \{\textbf{V-1-1}, (3,2), (9,n)\}$, where n is unknown

Lemma 23. *If* $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 2$, *and* $\sum_{i=1}^{17} x_i = 4$, *then all seventeen cats can be identified after five scans.*

Proof. The first scan is (x_9, x_{10}, x_{11}). It yields 0 or 1, since $\sum_{i=9}^{17} \leq 1$.

1. If $x_9 + x_{10} + x_{11} = 0$

   The endgame becomes $\{\text{V-1-1}, (3, 2), (6, n)\}$, thus four more scans suffice.

2. If $x_9 + x_{10} + x_{11} = 1$

   Since $x_3 = 0$ implies there are at least 5 copy cats, so we have $x_3 = 1$ and $x_{10} = x_{11} = 0$, the endgame becomes $\{(3, 2), (4, 1)\}$, three more scans suffice.

$\square$

5.24 $S = \{\textbf{V-1-1}, (12, n)\}$, where n is unknown

Lemma 24. *If $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$ and $\sum_{i=1}^{17} x_i = 4$, then all seventeen cats can be identified after seven scans.*

Proof. The first scan is (x_6, x_7, x_8), after which the endgame becomes $\{\text{V-1-1}, (3, 1), (9, n-1)\}$, $\{\text{V-1-1}, (3, 2), (9, n-2)\}$, or $\{\text{V-1-1}, (9, n)\}$, so at more six more scans suffice. $\square$

5.25 $S = \{\textbf{V-1-1}, (3, 1), (12, n)\}$, where n is unknown

Lemma 25. *If $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 1$, and $\sum_{i=1}^{20} x_i = 4$, then all twenty cats can be identified after seven scans.*

Proof. The first scan is (x_9, x_{10}, x_{11}).

1. If $x_9 + x_{10} + x_{11} = 3$

   This implies there are at least 5 copy cats, so it is impossible.

2. If $x_9 + x_{10} + x_{11} = 0$

   Now the endgame becomes $S = \{\text{V-1-1}, (3, 1), (9, n)\}$, six more scans suffice.

3. If $x_9 + x_{10} + x_{11} = 2$

   In this case all four copy cats are among the first eleven cats, so $x_3 = 1$ and the endgame is $\{(3, 1), (3, 2)\}$, three more scans suffice.

4. If $x_9 + x_{10} + x_{11} = 1$

   The second scan is (x_3, x_{12}, x_{13}). It cannot yield 3, otherwise there are more than four copy cats. So there are three cases:

   (a) If $x_3 + x_{12} + x_{13} = 2$

   Now all four copy cats are among the first thirteen cats, so $x_3 = x_{12} + x_{13} = 1$ and the endgame becomes $\{(3, 1), (3, 1), (2, 1)\}$, three more scans suffice.

(b) If $x_3 + x_{12} + x_{13} = 1$
If $x_3 = 0$, there are at least 5 copy cats, therefore $x_3 = 1$ and $x_1+x_2 = x_4 + x_5 = x_{12} + x_{13} = 0$, the endgame becomes $\{(3,1),(3,1),(7,1)\}$, five more scans suffice.

(c) If $x_3 + x_{12} + x_{13} = 0$
In this case the endgame is $\{(2,1),(2,1)(3,1),(3,1)\}$, four more scans suffice.

□

5.26 $S = \{$**V-1-1**$, (3,2), (12,n)\}$, where n is unknown

Lemma 26. *If* $x_1+x_2+x_3 = x_3+x_4+x_5 = 1$, $x_6+x_7+x_8 = 2$, *and* $\sum_{i=1}^{20} x_i = 4$, *then all twenty cats can be identified after six scans.*

Proof. The first scan is (x_9, x_{10}, x_{11}). It yields 0 or 1, since $\sum_{i=9}^{20} \leq 1$.

1. If $x_9 + x_{10} + x_{11} = 0$

   The endgame becomes $\{$V-1-1$, (3,2), (9,n)\}$, thus five more scans suffice.

2. If $x_9 + x_{10} + x_{11} = 1$

   In this case $x_3 = 0$ implies there are at least 5 copy cats, so we have $x_3 = 1$ and $x_{10} = x_{11} = 0$, the endgame becomes $\{(3,2),(7,1)\}$, four more scans suffice.

□

5.27 $S = \{$**V-1-1**$, (15,n)\}$, where n is unknown

Lemma 27. *If* $x_1 + x_2 + x_3 = x_3 + x_4 + x_5 = 1$ *and* $\sum_{i=1}^{20} x_i = 4$, *then all twenty cats can be identified after eight scans.*

Proof. The first scan is (x_6, x_7, x_8), after which the endgame becomes $\{$V-1-1$, (3,1), (12, n-1)\}$, $\{$V-1-1$, (3,2), (12, n-2)\}$, or $\{$V-1-1$, (12,n)\}$, so at more seven more scans suffice. □

6 Plain Strategy

Equipped with above tactics and endgames, we are going to solve the original problem: Identify all copy cats from a group of cats that contains 4 copy cats and 16 real cool cats.

In this section, we introduce a strategy proposed by Yen-Kang Fu. We call it *Plain Strategy* since it is straightforward and easy to understand. In most cases it can solve the problem after ten scans, but in the most complicated case it requires eleven scans.

We separate all 20 cats into seven subgroups: (x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8, x_9), (x_{10}, x_{11}, x_{12}), (x_{13}, x_{14}, x_{15}), (x_{16}, x_{17}, x_{18}), and (x_{19}, x_{20}). After six scans on the first six subgroups, we know the copy cat distribution among these seven subgroups.

There are three cases:

1. If it is 3-1, 2-2, or 2-1-1

   These endgames have been analysed. At most four more scans suffice.

2. If it is $\{(3,1),(3,1),(3,1),(2,1)\}$

   In this endgame four more scans also suffice.

3. If it is $\{(3,1),(3,1),(3,1),(3,1)\}$

   In this case five more scans suffice.

In summary, if the copy cat distribution is $\{(3,1),(3,1),(3,1),(3,1)\}$, the Plain Strategy needs 11 scans; otherwise, in all other cases, 10 scans suffice.

7 Improved Plain Strategy

The Improved Plain Strategy starts as the normal Plain Strategy, but it would slightly change its scan plan if there is a good chance that the 1-1-1-1 distribution would appear, so that it can save one valuable scan.

For the Improved Plain Strategy, the first scan is (x_1, x_2, x_3). If it yields 0, then the second scan is (x_4, x_5, x_6). If it still yields 0, then the third scan is (x_7, x_8, x_9).

There are four cases:

7.1 If $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = 0$

If all first three scans yield 0, then all 4 copy cats are among the remaining 11 cats. Now the most complicated case, $\{(3,1),(3,1),(3,1),(3,1)\}$, has been avoided, so the normal Plain Strategy works.

7.2 If $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$ and $x_7 + x_8 + x_9 > 0$

If $x_7 + x_8 + x_9 = 2$ or 3, then the 1-1-1-1 distribution has been avoided and we can continue as the normal Plain Strategy. Therefore we only need to consider the case $x_7 + x_8 + x_9 = 1$.

Here the fourth scan is (x_9, x_{10}, x_{11}). If it yields 3 or 0, then $\{(3,1),(3,1),(3,1),(3,1)\}$ has been avoided and the normal Plain Strategy works.

So there are two more cases:

1. If $x_9 + x_{10} + x_{11} = 2$

   The five cat $(x_7, x_8, x_9, x_{10}, x_{11})$ form V-1-2, so after the fifth scan on (x_7, x_8, x_{10}), we are able to identify all values of x_9, x_{10}, and x_{11}, as well as the value of $x_7 + x_8$, which is either 0 or 1.

   (a) If $x_7 + x_8 = 1$
   The last copy cat is among the remaining 9 cats and the endgame is $\{(2,1),(9,1)\}$, so four more scans suffice.

   (b) If $x_7 + x_8 = 0$
   The last two copy cats are among the remaining 9 cats and the endgame is $\{(9,2)\}$, so five more scans suffice.

2. If $x_9 + x_{10} + x_{11} = 1$

   Now the endgame becomes $\{\text{V-1-1}, (9,n)\}$, so six more scans suffice.

7.3 If $x_1 = x_2 = x_3 = 0$ and $x_4 + x_5 + x_6 > 0$

Using the same argument in the previous case, we only need to consider the case $x_4 + x_5 + x_6 = 1$ and $x_6 + x_7 + x_8 \in \{1, 2\}$. Three scans have been used so far.

1. If $x_6 + x_7 + x_8 = 2$

   The five cats $(x_4, x_5, x_6, x_7, x_8)$ form V-1-2, so after the fourth scan on (x_4, x_5, x_7), we are able to identify all values of x_6, x_7, and x_8, as well as the value of $x_4 + x_5$, which is either 0 or 1.

   (a) If $x_4 + x_5 = 1$
   The last copy cat is among the remaining 12 cats and the endgame is $\{(2,1),(12,1)\}$, so five more scans suffice.

   (b) If $x_4 + x_5 = 0$
   The last two copy cats are among the remaining 12 cats and the endgame is $\{(12,2)\}$, so six more scans suffice.

2. If $x_6 + x_7 + x_8 = 1$

   Now the endgame becomes $\{\text{V-1-1}, (12,n)\}$, so seven more scans suffice.

7.4 If $x_1 + x_2 + x_3 > 0$

For the same reasons in the previous two cases, we only need to consider the case $x_1 + x_2 + x_3 = 1$ and $x_3 + x_4 + x_5 \in \{1, 2\}$. Two scans have been used so far.

1. If $x_3 + x_4 + x_5 = 2$

   The five cats $(x_1, x_2, x_3, x_4, x_5)$ form V-1-2, so after the third scan on (x_1, x_2, x_4), we are able to identify all values of x_3, x_4, and x_5, as well as the value of $x_1 + x_2$, which is either 0 or 1.

(a) If $x_1 + x_2 = 1$

The last copy cat is among the remaining 15 cats and the endgame is $\{(2, 1), (15, 1)\}$, so six more scans suffice.

(b) If $x_1 + x_2 = 0$

The last two copy cats are among the remaining 15 cats and the endgame is $\{(15, 2)\}$, so seven more scans suffice.

2. If $x_3 + x_4 + x_5 = 1$

Now the endgame becomes $\{\text{V-1-1}, (15, n)\}$, so eight more scans suffice.

8 Conclusion

Using the Improved Plain Strategy, we are able to identify all 4 copy cats from 20 cats after at most 10 scans. This proves that $f(20, 4, 3) \leq 10$.