A Semidefinite Programming Approach to the Anaylsis of Functional Differential Equations

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Research Overview:

Stability of Differential Equations with Delay

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Convex Optimization

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Sum-of-Squares

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Positive Operators

$\Downarrow$

Results and Examples

Consider: A System of Linear Differential Equations with Discrete Delays

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i)$$

Problem: Stability

Given specific $A_i \in \mathbb{R}^{n \times n}$ and $\tau_i \in \mathbb{R}^+$,
and arbitrary initial condition x_0,
does $\lim_{t \to \infty} x(t) = 0$?

Example: Standard Test Case

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

Determine the minimum and maximum stable τ.

	$\tau_{\min}$	$\tau_{\max}$
Numeric	.10017	1.7172
Analytic	.10017	1.71785

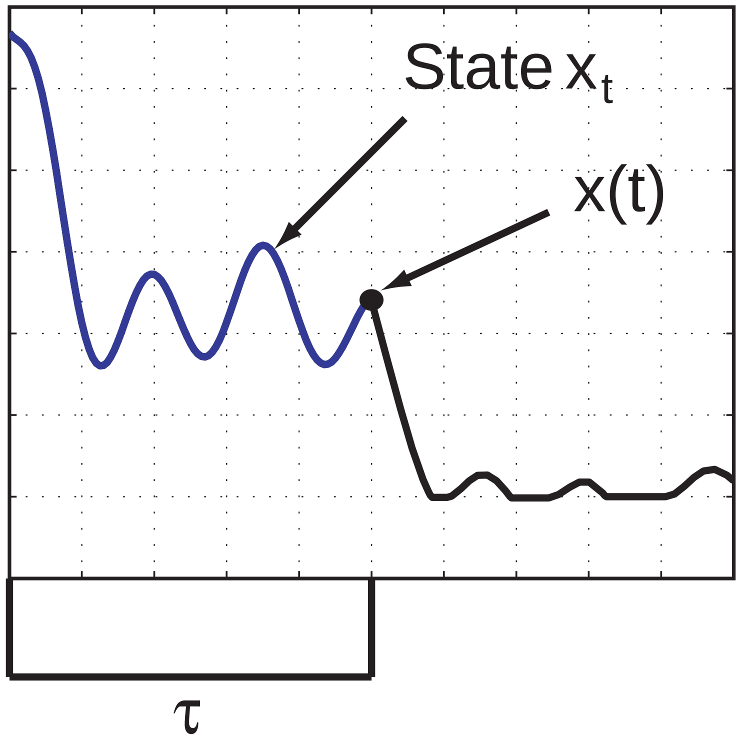
Table 1: τ_{max} and τ_{min}

Consider: The Class of Functional Differential Equations

For a given functional $f : \mathcal{C}_\tau \to \mathbb{R}^n$, f defines a system of functional differential equations:

$$\dot{x}(t) = f(x_t)$$

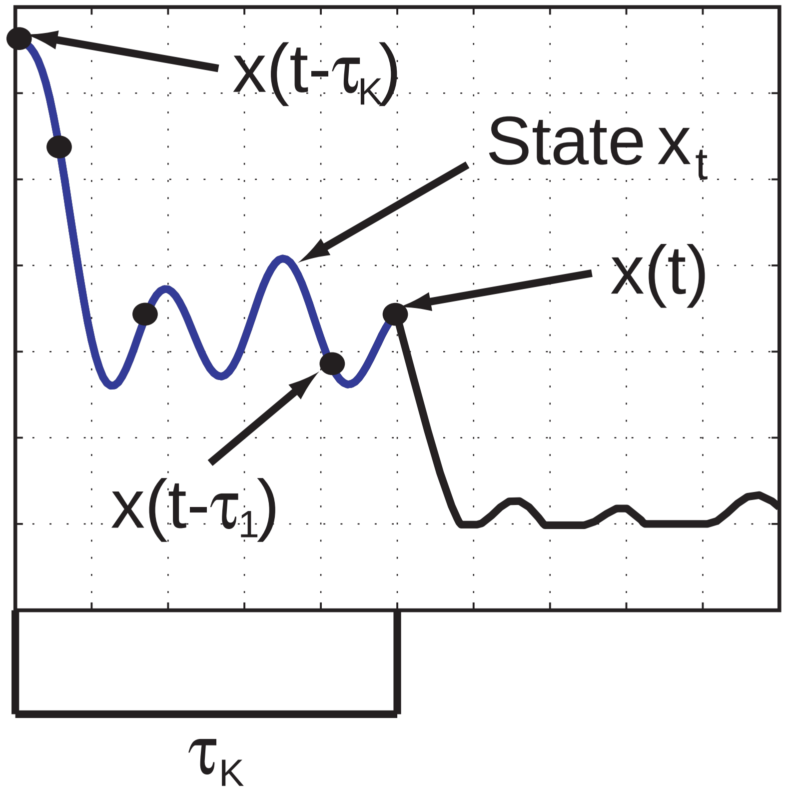
$$x_t(\theta) := x(t+\theta) \qquad \theta \in [-\tau, 0]$$



- We call $x_t \in \mathcal{C}_\tau$ the full state of the system at time t.
- $x(t)$ is the present state of the system at time t.

A Specific Case: Differential Systems with a Delay in the State

$$\dot{x}(t) = g\big(x(t), x(t-\tau_1), \ldots, x(t-\tau_K)\big)$$



Lyapunov Functions: A Method of Stability Analysis

Lyapunov functions can be used to prove stability of functional differential equations.

Theorem 1 *A functional differential equation is stable if there exists a* $V : \mathcal{C}_\tau \to \mathbb{R}$ *and* $\epsilon > 0$ *such that for all* $x_t \in \mathcal{C}_\tau$*, we have*

$$V(x_t) \geq \epsilon \|x(t)\|_2$$
$$\dot{V}(x_t) \leq 0.$$

$\dot{V}$ is the Lie derivative.

Aside:

The set of positive functionals is convex.
The set of negative functionals is convex.
$\Rightarrow$ If the map $V \mapsto \dot{V}$ is linear, then stability analysis is convex optimization...

more to come.

Linear Systems Have: Quadratic Lyapunov functions

Suppose f is linear and defines a stable system.

Then in most cases there exists some positive linear map $A : \mathcal{C}_\tau \to \mathcal{C}_\tau$ such that the Lyapunov function

$$V(x_t) = \langle x_t, Ax_t \rangle$$

proves stability of the system.

- For linear systems with delay,

$$\dot{V}(x_t) = \langle x_t, Bx_t \rangle$$

and the map $A \mapsto B$ is linear.

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Results and Examples

Computational Complexity: Is it NP-Hard?

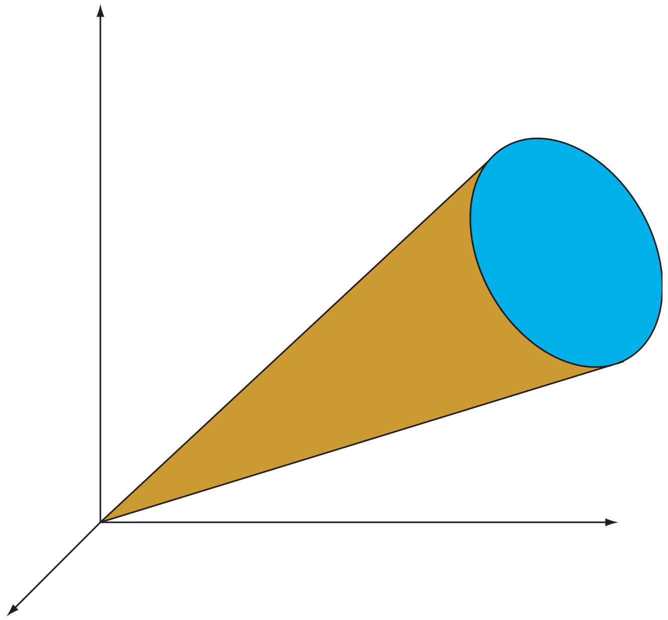
Problems in P:

- The shortest path
- Stability of linear systems in finite dimensions
- Linear Programming
- Semidefinite programming?

Problems in NP+:

- The traveling salesman
- Matrix Copositivity
- Positivity of Polynomials
- μ
- Delay-Independent Stability

Convex Optimization



Problem:

$$\max \; bx$$
$$\text{subject to} \quad Ax \in C$$

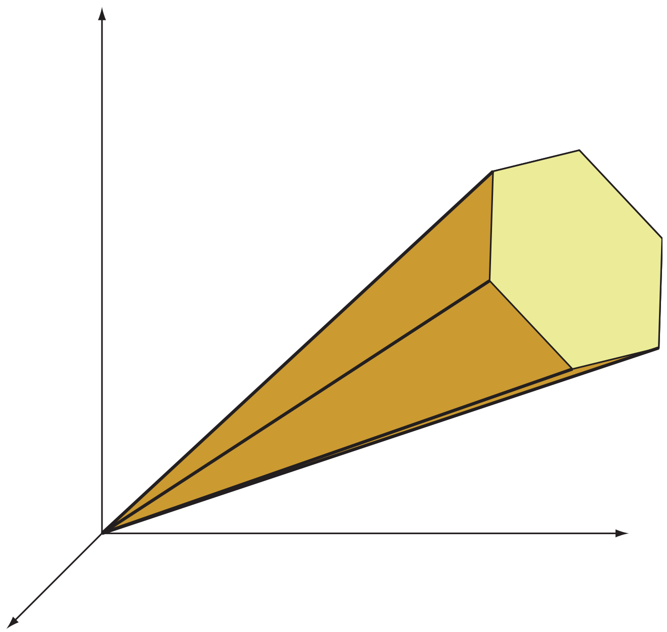
The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Computational Tractability: Convex Optimization over C is, in general, tractable if

- There is an efficient set membership test for $x \in C$

Semidefinite Programming(SDP)



Problem:

$$\max \ b^T x$$

subject to $$A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0$$

Here

- $x \in \mathbb{R}^m$ and the A_i are symmetric matrices.
- $\succeq 0$ denotes membership in the cone of positive semidefinite matrices.

Computationally Tractable

Semidefinite Programming(SDP): Common Examples in Control

- Stability

$$A^T X + XP \prec 0$$
$$X \succ 0$$

- Stabilization

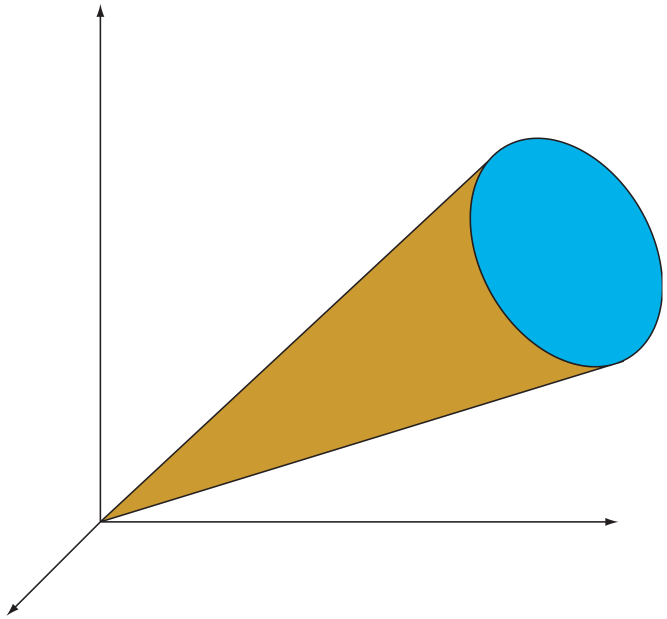
$$AX + BZ + XA^T + Z^T B^T \prec 0$$
$$X \succ 0$$

- H_2 Synthesis

$$\min Tr(W)$$
$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T \prec 0$$
$$\begin{bmatrix} X & (CX + DZ)^T \\ (CX + DZ) & W \end{bmatrix} \succ 0$$

- KYP Lemma

Polynomial Programming



Problem:

$$\max \; c^T x$$

subject to $A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$

The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha} \, y^\alpha$$

Computationally Intractable

Polynomial Programming: Examples

- Stability of Nonlinear Systems

$$f(y)^T \nabla p(y) < 0$$
$$p(y) > 0$$

- Matrix Copositivity

$$y^T M y - g(y)^T y \geq 0$$
$$g(y) \geq 0$$

- Integer Programming

$$\max \gamma$$
$$p_0(y)(\gamma - f(y)) - (\gamma - f(y))^2 + \sum_{i=1}^n p_i(y)(y_i^2 - 1) \geq 0$$
$$p_0(y) \geq 0$$

- Also μ

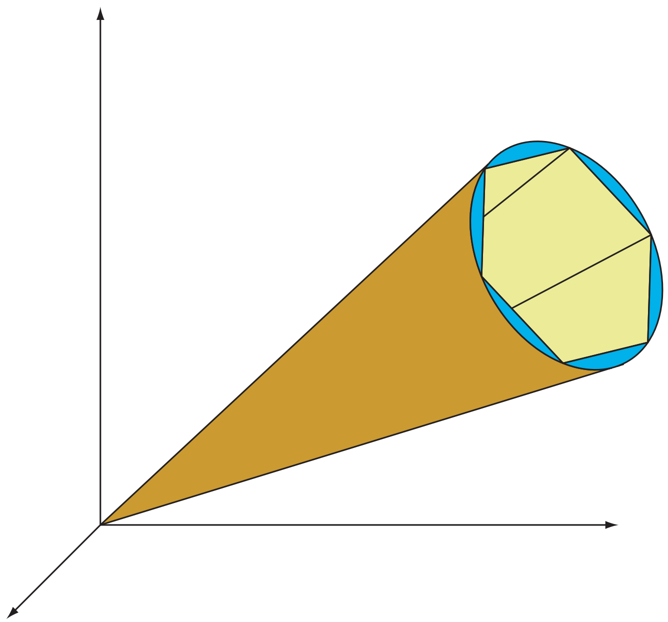
Positivstellensatz results are commonly used to set up these problems.

Sum-of-Squares(SOS) Programming

Problem:

$$\max \; c^T x$$

subject to $\quad A_0(y) + \sum_i^n x_i A_i(y) \in \Sigma_s$



- Σ_s is the cone of *sum-of-squares* matrices. If $S \in \Sigma_s$, then for some $G_i \in \mathbb{R}[x]$,

$$S(y) = \sum_{i=1}^r G_i(y)^T G_i(y)$$

Computationally Tractable: $S \in \Sigma_s$ is an SDP constraint.

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SOS Programming: Why is $M \in \Sigma_s$ an SDP?

Define $Z_d(x)$ to be the vector of monomial bases in dimension n of degree d or less.

For example, if $n = 1$, and $x \in \mathbb{R}^2$, then

$$Z_2(x)^T = \begin{bmatrix}1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2\end{bmatrix}$$

If $n = 2$, and $x \in \mathbb{R}^2$, then

$$Z_1(x)^T = \begin{bmatrix}1 & x_1 & x_2 & & & \\ & & & 1 & x_1 & x_2\end{bmatrix}$$

Lemma 1 *Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ iff there exists some $Q \succeq 0$ such that*

$$M(x) = Z_d(x)^T Q Z_d(x).$$

Note: Sometimes we won't mention d explicitly.

SOS Programming: Example

$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

Problem: Is $M \in \Sigma_s$?

Algorithm:

Step 1: Write

$$M(y,z) = NZ_4(y,z)$$

Step 2: Construct B such that if $N = B\,\text{vec}(Q)$, then

$$NZ_4(y,z) = Z_2(y,z)^T Q Z_2(y,z)$$

This only depends on Z_2 and Z_4

Step 3: Find $Q \succ 0$ such that $N = B\,\text{vec}(Q)$

SOS Programming: Solution

$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

$$\begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}$$

$$= \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix}^T \begin{bmatrix} yz & 1-y \\ z & y^2 \end{bmatrix} \in \Sigma_s$$

The First Step: How to Construct SOS Programs?

Few questions are naturally expressed as polynomial programs.

- Instead consider optimization over semialgebraic sets

$$\max f(x) :$$
$$p_i(x) \geq 0$$
$$q_i(x) = 0$$

Special cases include:

- Matrix Copositivity:

$$\min x^T M x :$$
$$x \geq 0$$

- Integer programming:

$$\max f(x) :$$
$$x_i^2 = 1$$

The Next Step: Positivstellensatz

Let

$$\mathcal{P} := \left\{ x : \begin{matrix} p_i(x) \geq 0 & i = 1, \dots, k \\ q_j(x) = 0 & j = 1, \dots, m \end{matrix} \right\}$$

Theorem 2 (Putinar) *Suppose* $\mathcal{P}$ *is "compact+". Suppose* $f(x) \geq 1$ *for* $x \in \mathcal{P}$*. Then there exist* $s_i \in \Sigma_s$ *and* $t_i \in \mathbb{R}[x]$ *such that*

$$f(x) - \sum_{i=1}^{k} s_i(x) p_i(x) + \sum_{i=1}^{m} t_i(x) q_i(x) \in \Sigma_s$$

There are many other formulations

Example: Robust Lyapunov Stability

Problem: Is $\dot{x}(t) = f(\alpha, x(t))$ stable for $\alpha \in \Delta := \{\alpha : \|\alpha\|^2 < 1\}$?

$$\begin{aligned} &\text{find } V : \quad \text{for any } \alpha \in \Delta, \\ &V(x) > 0 \\ &\dot{V}(x) < 0 \end{aligned}$$

Equivalently:

$$\begin{aligned} &\text{find } V(x) = \sum_{\alpha_i} c_i x^{\alpha_i} : \\ &f(\alpha, x)^T \nabla V(x) < 0 \quad \text{for } \alpha \in \Delta \\ &V(x) > \epsilon \|x\|^2 \end{aligned}$$

SOS Program:

$$\begin{aligned} &\text{find } V : \\ &-f(\alpha, x)^T \nabla_x V(x) - s(\alpha, x)(\|\alpha\|^2 - 1) \in \Sigma_s \\ &V(x) - \epsilon \|x\|^2 \in \Sigma_s, \qquad s(\alpha, x) \in \Sigma_s \end{aligned}$$

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Results and Examples

Return to Linear Differential Equations with Delay:

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i)$$

Stable iff $\exists\, V > 0 : \dot{V} < 0$, where

$$V(x) = \int_{-\tau_m}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds + \int_{-\tau_m}^{0} \int_{-\tau_m}^{0} x(s) N(s,t) x(t) ds dt$$

Problem: Find M and N so that:

$$V(x) > 0$$
$$\dot{V}(x) < 0$$

Result: Positivity of Part 1

Theorem 3 *Let M be piecewise-continuous, then following are equivalent*

1. *There exists some $\epsilon > 0$ so that*

$$\int_{-\tau_m}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds \geq \epsilon \|x\|^2$$

2. *There exists a function T and $\epsilon' > 0$ such that*

$$\int_{-\tau_m}^{0} T(s) ds = 0 \quad \textit{and} \quad M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon' I$$

Computationally Tractable:

- Assume M and T are polynomials
- The constraint $\int_{-\tau_m}^{0} T(s) ds = 0$ is linear
- For the 1-D case, Σ_s is exact.

Example: Positive Multipliers

$$M(s) = \begin{bmatrix} -2s^2+2 & s^3-s \\ s^3-s & s^4+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} 3s^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s & s^2 \\ 1 & -s \end{bmatrix}^T \begin{bmatrix} s & s^2 \\ 1 & -s \end{bmatrix} + \begin{bmatrix} 3s^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

And

$$\begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \int_{-1}^{0} (3s^2-1)ds = 0$$

Result: Positivity of Part 2

Theorem 4 *Suppose $N(s,t)$ is a polynomial. Then the following are equivalent:*

- $$\int_{-\tau_m}^{0}\int_{-\tau_m}^{0} x(s)^T N(s,t) x(t) ds dt \geq 0 \qquad \textit{for all } x \in \mathcal{C}$$
- *There exists a $Q \geq 0$ such that*
$$N(s,t) + N(t,s)^T = Z(s)^T Q Z(t)$$

Notes:

- Map is affine
- N is separable

Example: Positive Integral Operators

If

$$N(s,t) = \begin{bmatrix} 1-t-s+2st & 1-s-st^2 \\ 1-t-s^2t & 1+s^2t^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 2 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 0 \\ 0 & 1 \\ 0 & t^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 0 \\ 0 & 1 \\ 0 & t^2 \end{bmatrix} = \begin{bmatrix} 1-s & 1 \\ -s & s^2 \end{bmatrix}^T \begin{bmatrix} 1-t & 1 \\ -t & t^2 \end{bmatrix}$$

Then

$$\int_{-\tau}^{0}\int_{-\tau}^{0} x(s)^T N(s,t) x(t) ds dt = \int_{-\tau}^{0}\int_{-\tau}^{0} x(s)^T G(s)^T G(t) x(t) ds dt$$

$$= \int_{-\tau}^{0} x(s)^T G(s)^T ds \int_{-\tau}^{0} G(t) x(t) dt = K^T K \geq 0$$

Result: Positivity of Part 2 Continued

Lemma 2 *Suppose*

$$Q(s) \geq 0$$

Now Define

$$\begin{bmatrix} N_{11}(t,s,\theta) & N_{12}(t,s,\theta) \\ N_{12}(t,s,\theta)^T & N_{22}(t,s,\theta) \end{bmatrix} = Z(t)^T Q(s) Z(\theta)$$

Let

$$k_1(t,\theta) = \int_0^\theta N_{22}(t,s,\theta)ds + \int_\theta^t N_{12}(t,s,\theta)^T ds + \int_t^1 N_{11}(t,s,\theta)ds$$

$$k_2(t,\theta) = \int_0^t N_{22}(t,s,\theta)ds + \int_t^\theta N_{12}(t,s,\theta)ds + \int_\theta^1 N_{11}(t,s,\theta)ds$$

and define

$$k(t,s) = \begin{cases} k_1(t,s) & 0 \leq s < t \leq 1 \\ k_2(t,s) & 0 \leq t < s \leq 1 \end{cases}$$

Then for any $x \in \mathcal{C}$

$$\int_0^1 \int_0^1 x(s)^T k(s,t) x(t) ds dt \geq 0$$

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Example: Standard Test Case 2 - Multiple Delays

We now consider a system with multiple delays.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20} x(t - \frac{\tau}{2}) + \frac{19}{20} x(t - \tau) \right]$$

A bisection method was used and results are listed below.

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 2: τ_{max} and τ_{min} using a piecewise-linear functional and our approach and compared to the analytical limit.

Parametric Uncertainty

Result: We can construct parameter-dependent Lyapunov functionals.

Approach: We replace the semidefinite programming constraint

$$Q \succeq 0$$

with the SOS programming constraint

$$Q(\alpha) \in \Sigma_s.$$

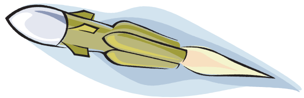
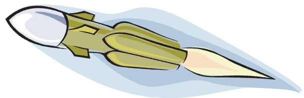
Example: Standard Test Case 1 Revisited

By including τ as an uncertain parameter in the Lyapunov functionals, we can prove stability over the interval $[\tau_{\min}, \tau_{\max}]$ directly.

d in τ	d in θ	$\tau_{\min}$	$\tau_{\max}$
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 3: Stability on the interval $[\tau_{\min}, \tau_{\max}]$ vs. degree using a parameter-dependent functional

Example: Remote Control


A Simple Inertial System: Suppose we are given a specific type of PD controller that we want to implement.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

The controller is stable for all positive a. Now suppose we want to maintain control from a remote location. When we include the **communication delay**, the equation becomes.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable. The model is linear, but contains a parameter and an uncertain delay.

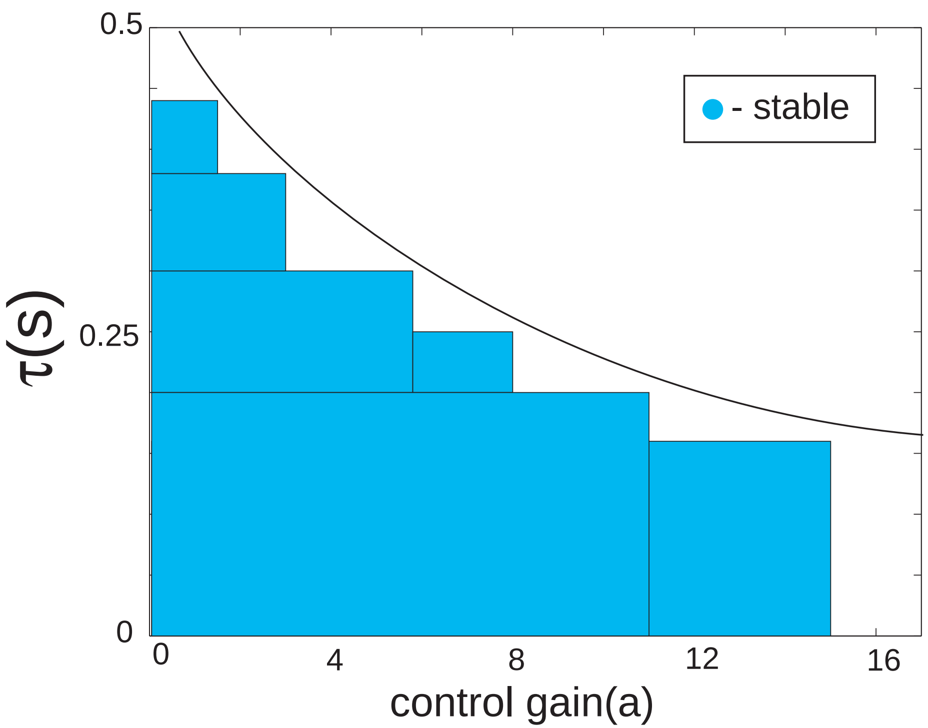
Example: Remote Control

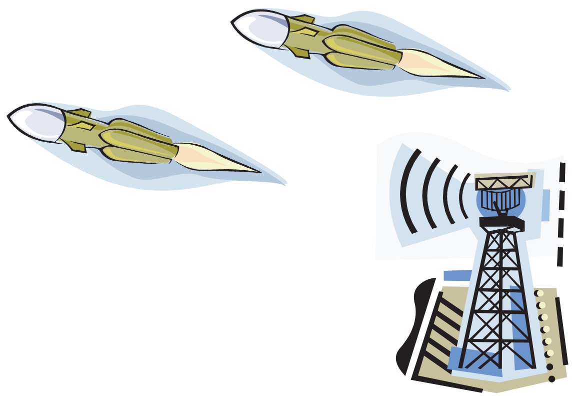
Recall that we considered an inertial system controlled remotely using PD control

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable?

- We use parameter-dependent functionals.





Result:

- An new approach to solving the Lyapunov inequality

Practical Impact:

- Linear with Time-Delay
  - Numerically well-conditioned and convergent
  - We can show that relatively large linear time-delay systems are stable
- Uncertain with Time-Delay
  - We can prove stability over ranges of operating conditions
- Nonlinear with Time-Delay
  - Provides an easy way of testing stability of very complicated systems

Research Directions

Theory

- Stabilizing Controllers
- Partial Differential Equations
- Optimal Controller Synthesis
- The KYP lemma

Applications

Industrial and Electrical:

- Communication Systems
- Manufacturing

Biological:

- Cancer Therapy
- HIV Therapy

Semi-Algebraic Sets

Recall the general optimization problem:

$$\max f(x) :$$
$$p_i(x) \geq 0$$
$$q_i(x) = 0$$

Reformulate the problem using semi-algebraic sets.

$$\min \gamma :$$
$$\mathcal{P} = \emptyset$$
$$\mathcal{P} := \{x : f(x) - \gamma > 0, p_i(x) \geq 0, q_i(x) = 0\}$$

Example: Integer Programming

$$\min \gamma :$$
$$\mathcal{P} = \emptyset$$
$$\mathcal{P} := \{x : f(x) - \gamma > 0, x_i^2 - 1 = 0\}$$

The Next Step: Positivstellensatz

Theorem 5 (Stengle) *The following are equivalent*

- $$\left\{x : \begin{matrix} p_i(x) \geq 0 & i = 1, \ldots, k \\ q_j(x) = 0 & j = 1, \ldots, m \end{matrix}\right\} = \emptyset$$

- *There exist* $t_i \in \mathbb{R}[x]$, $s_i, r_{ij}, \ldots \in \Sigma_s$ *such that*

$$-1 = \sum_i q_i t_i + s_0 + \sum_i s_i p_i + \sum_{i \neq j} r_{ij} p_i p_j + \cdots$$

Bonus Material: Nonlinear Time-delay systems

Consider nonlinear systems which have a single delay.

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K))$$

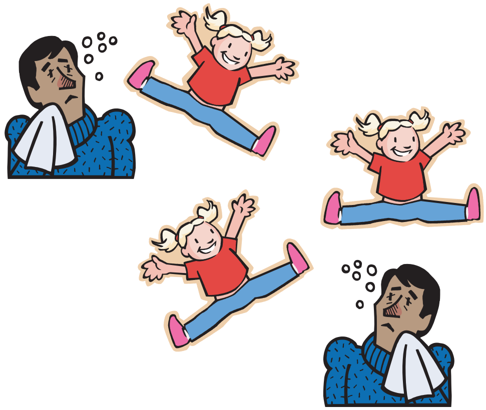
Here we assume $x(t) \in \mathbb{R}^n$ and f is polynomial.

We use a generalization of the compete quadratic functional of the following form.

$$\begin{aligned} V(\phi) :&= \int_{-\tau_K}^0 f_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau_K}^0 \int_{-\tau_K}^0 f_2(\phi(\theta), \phi(\omega), \theta, \omega) d\theta d\omega \\ &= \int_{-\tau_K}^0 Z(\phi(0), \phi(\theta))^T M(\theta) Z(\phi(0), \phi(\theta)) d\theta \\ &+ \int_{-\tau_K}^0 \int_{-\tau_K}^0 Z(\phi(\theta))^T R(\theta, \omega) Z(\phi(\omega)) d\theta d\omega \end{aligned}$$

Computation: We represent M and R using results generalized from the linear case.

Example: Epidemiological Model of Infection



Consider a human population subject to non-lethal infection by a cold virus. The disease has **incubation period** (τ). Cooke(1978) models the percentage of infected humans(y) using the following equation.

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Where

- a is the rate of recovery for infected persons
- b is the rate of infection for exposed people

The model is nonlinear and contains delay. Equilibria exist at $y^* = 0$ and $y^* = (b-a)/b$.

Example: Epidemiological Model

Recall the dynamics of infection are given by

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Cooke used the following Lyapunov functional to prove delay-independent stability of the 0 equilibrium for $a > b > 0$.

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{1}{2}\int_{-\tau}^{0} a\phi(\theta)^2 d\theta$$

Using semidefinite programming, we were also able to prove delay-independent stability for $a > b > 0$ using the following functional.

$$V(\phi) = 1.75\phi(0)^2 + \int_{-\tau}^{0}(1.47a + .28b)\phi(\theta)^2 d\theta$$

Conclusion: When the rate of recovery is greater than the rate of infection, the epidemic will die out.