Information Gain

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Bits

You are watching a set of independent random samples of X

You see that X has four possible values

P(X=A) = 1/4	P(X=B) = 1/4	P(X=C) = 1/4	P(X=D) = 1/4

So you might see: BAACBADCDADDDA…

You transmit data over a binary serial link. You can encode each reading with two bits (e.g. A = 00, B = 01, C = 10, D = 11)

0100001001001110110011111100…

Fewer Bits

Someone tells you that the probabilities are not equal

$P(X=A) = 1/2$	$P(X=B) = 1/4$	$P(X=C) = 1/8$	$P(X=D) = 1/8$

It's possible…

…to invent a coding for your transmission that only uses 1.75 bits on average per symbol. How?

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It's possible…

…to invent a coding for your transmission that only uses 1.75 bits on average per symbol. How?

A	0
B	10
C	110
D	111

(This is just one of several ways)

Fewer Bits

Suppose there are three equally likely values…

P(X=A) = 1/3	P(X=B) = 1/3	P(X=C) = 1/3

Here’s a naïve coding, costing 2 bits per symbol

A	00
B	01
C	10

Can you think of a coding that would need only 1.6 bits per symbol on average?

In theory, it can in fact be done with 1.58496 bits per symbol.

General Case

Suppose X can have one of *m* values… $V_1, V_2, \ldots V_m$

$P(X=V_1) = p_1$	$P(X=V_2) = p_2$	….	$P(X=V_m) = p_m$

What's the smallest possible number of bits, on average, per symbol, needed to transmit a stream of symbols drawn from X's distribution? It's

$$H(X) = -p_1 \log_2 p_1 - p_2 \log_2 p_2 - \ldots - p_m \log_2 p_m$$

$$= -\sum_{j=1}^{m} p_j \log_2 p_j$$

H(X) = The entropy of X

- "High Entropy" means X is from a uniform (boring) distribution
- "Low Entropy" means X is from varied (peaks and valleys) distribution

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symbol, … tream … m X's
distributi…

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A histogram of the frequency distribution of values of X would be flat

A histogram of the frequency distribution of values of X would have many lows and one or two highs

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A histogram of the frequency distribution of values of X would be flat

..and so the values sampled from it would be all over the place

A histogram of the frequency distribution of values of X would have many lows and one or two highs

..and so the values sampled from it would be more predictable

H(X) = The entropy of X

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Entropy in a nut-shell

Low Entropy

High Entropy

Entropy in a nut-shell

Low Entropy

..the values (locations of soup) sampled entirely from within the soup bowl

High Entropy

..the values (locations of soup) unpredictable... almost uniformly sampled throughout our dining room

Specific Conditional Entropy H(Y|X=v)

Suppose I'm trying to predict output Y and I have input X

X = College Major

Y = Likes "Gladiator"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Let's assume this reflects the true probabilities

E.G. From this data we estimate

- *P(LikeG = Yes) = 0.5*
- *P(Major = Math & LikeG = No) = 0.25*
- *P(Major = Math) = 0.5*
- *P(LikeG = Yes | Major = History) = 0*

Note:

- *H(X) = 1.5*
- *H(Y) = 1*

Specific Conditional Entropy H(Y|X=v)

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History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

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Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Example:

- $H(Y|X=Math) = 1$
- $H(Y|X=History) = 0$
- $H(Y|X=CS) = 0$

Conditional Entropy H(Y|X)

X = College Major

Y = Likes "Gladiator"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Conditional Entropy:

$H(Y|X)$ = The average specific conditional entropy of Y

= if you choose a record at random what will be the conditional entropy of Y, conditioned on that row's value of X

= Expected number of bits to transmit Y if both sides will know the value of X

$$= \Sigma_j \, Prob(X=v_j) \, H(Y \mid X = v_j)$$

Conditional Entropy

X = College Major

Y = Likes "Gladiator"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Conditional Entropy:

$H(Y|X)$ = The average conditional entropy of Y

$= \Sigma_j Prob(X=v_j)\ H(Y \mid X = v_j)$

Example:

v_j	$Prob(X=v_j)$	$H(Y \mid X = v_j)$
Math	0.5	1
History	0.25	0
CS	0.25	0

$H(Y|X) = 0.5 * 1 + 0.25 * 0 + 0.25 * 0 = 0.5$

Information Gain

X = College Major

Y = Likes "Gladiator"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Information Gain:

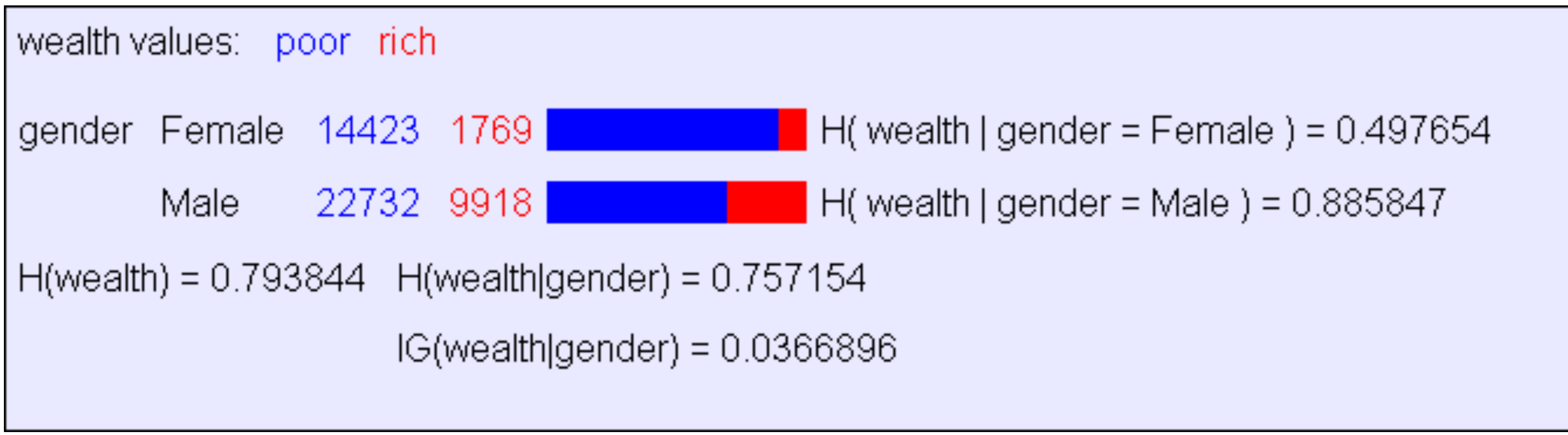
$IG(Y|X)$ = **I must transmit** Y. **How many bits on average would it save me if both ends of the line knew** X**?**

$$IG(Y|X) = H(Y) - H(Y|X)$$

Example:

- **H(Y) = 1**
- **H(Y|X) = 0.5**
- **Thus IG(Y|X) = 1 – 0.5 = 0.5**

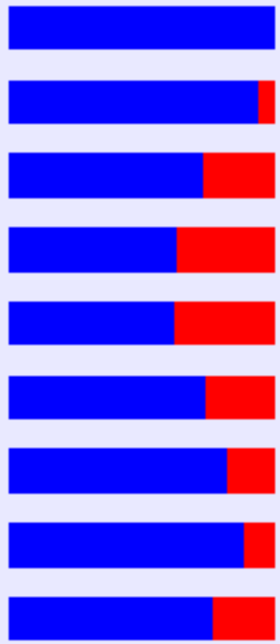
Information Gain Example



Another example

wealth values: poor rich

		poor	rich	
agegroup	10s	2507	3	H(wealth | agegroup = 10s) = 0.0133271
	20s	11262	743	H(wealth | agegroup = 20s) = 0.334906
	30s	9468	3461	H(wealth | agegroup = 30s) = 0.838134
	40s	6738	3986	H(wealth | agegroup = 40s) = 0.951961
	50s	4110	2509	H(wealth | agegroup = 50s) = 0.957376
	60s	2245	809	H(wealth | agegroup = 60s) = 0.834049
	70s	668	147	H(wealth | agegroup = 70s) = 0.680882
	80s	115	16	H(wealth | agegroup = 80s) = 0.535474
	90s	42	13	H(wealth | agegroup = 90s) = 0.788941



H(wealth) = 0.793844 H(wealth|agegroup) = 0.709463

IG(wealth|agegroup) = 0.0843813

Relative Information Gain

X = College Major

Y = Likes "Gladiator"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Relative Information Gain:

$RIG(Y|X)$ = **I must transmit** Y, **what fraction of the bits on average would it save me if both ends of the line knew** X**?**

$RIG(Y|X)$ **=** $H(Y) - H(Y \mid X) / H(Y)$

Example:

- $H(Y|X) = 0.5$
- $H(Y) = 1$
- ***Thus*** $IG(Y|X) = (1 - 0.5)/1 = 0.5$

What is Information Gain used for?

Suppose you are trying to predict whether someone is going live past 80 years. From historical data you might find…

- IG(LongLife | HairColor) = 0.01
- IG(LongLife | Smoker) = 0.2
- IG(LongLife | Gender) = 0.25
- IG(LongLife | LastDigitOfSSN) = 0.00001

IG tells you how interesting a 2-d contingency table is going to be.