Theorem 1 *If there is only one route, then it is best to paddle most quickly the segments where the downstream flow is fastest.*

More precisely: The optimal plan for paddling satisfies the following constraint. Let B be the flow speed and V be the paddling speed at one time, and let b be the flow speed and v be the paddling speed at a different time. If $B > b$ then $V \geq v$. That is, the constraint says to paddle faster on the faster downstream flow.

Proof: of the contrapositive:

If you have a plan P in which the paddler is padding at speed V during a time period with flow speed b and at speed v in a region with flow speed B, then there is an alternative plan Q which is faster than P.

Note that any plan that involves flowing backward for a certain length of time can be improved by deleting that part of the plan. So we can assume that $V > v > B > b$. That is, even at the slower paddling speed v on the faster part of the stream B, the paddler makes progress.

We construct the alternative plan Q as follows. Let G be the segment of the river in which, in P, the flow speed is B and the paddle speed is v and let H be the segment in which the flow speed is b and the paddle speed is V. Let $|G|$ and $|H|$ be the length of those segments. Thus in plan P the rower will spend time $|G|/(v-B)$ in G and $|H|/(V-b)$ in H.

Choose a small length of time $\epsilon > 0$ (specifically, $\epsilon < \min(|G|, |H|)/(V-b)$). Let x be a segment in G of length $\epsilon(V-B)$ and let X be a segment in H of length $\epsilon(V-b)$. Notice that X uses the higher velocity V as well. Thus x and X are both segments whose length can be traversed by the paddler at velocity V for time ϵ.

Plan Q is identical to plan P, except that, while the canoe is in segment x, the paddler will paddle with speed V and while the canoe is in segment X, the paddler will paddle with speed v.

So the time that plans P and Q spend everywhere outside x and X is identical. In plan Q, the total time that the paddler spends paddling at the higher speed V has increased in x by time $|x|/(V-B) = \epsilon$ and has decreased in X by time $|X|/(V-b) = \epsilon$, so it has remained constant. The time that the paddler spends paddling at the lower speed v has decreased in x by time $|x|/(v-B) = \epsilon(V-B)/(v-B)$ and has increased in X by $|X|/(v-b) = \epsilon(V-b)/(v-b)$, so all in all it has decreased by

$$\epsilon\left[\frac{V-B}{v-B} - \frac{V-b}{v-b}\right] = \epsilon\frac{(V-B)(v-b)-(V-b)(v-B)}{(v-b)(v-B)} = \epsilon\frac{BV+vb-(bV+Bv)}{(v-B)(v-b)} = \epsilon\frac{(V-v)(B-b)}{(v-B)(v-b)} > 0$$

Thus, the full ground is still covered; the time spent paddling at speed v has remained constant; the time spent peddling at speed V has decreased; so all in all the time has decreased. ▮