Another look at e[*]

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Abstract

This note describes a way of obtaining e that differs from the standard one. It could be used as an alternate way of showing how the value of e is obtained. No attempt is made to show the existence of the limit in the definition of e that appears in the final equation.

1. Introduction.

In this section we give an approximation of e using a technique we generalize in the next section.

If $f'(x)$, the derivative of $f(x)$, exists at point x, and you start at point x and move a distance Δx, the value at the point $x + \Delta$ is given by

$$f(x + \Delta x) \cong f(x) + f'(x) \cdot \Delta x \tag{1}$$

We want to find a constant, let's call it e, such that when it's raised to the power x obtaining the function e^x, the function's derivative is also e^x[‡].

Since $f'(x)$ equals $f(x)$, we rewrite equation (1) as

$$f(x + \Delta x) \cong f(x)(1 + \Delta x) \tag{2}$$

We will analyse this in the interval [1,2]. Let's take x = 1 and $\Delta x = 0.1$. So $x + \Delta x$ is 1.1. Equation (2) gives

$$f(1.1) \cong f(1)(1 + 0.1) \tag{3}$$

or

$$e^{1.1} \cong 1.1e \tag{4}$$

.

Now take $x = 1.1$ and use the same value of Δx, i.e., 0.1. We will be using the same increment in x in this and all subsequent steps since eventually we will let Δx approach zero. Continuing in this way

$$f(1.1 + 0.1) \cong 1.1f(1.1) \tag{5}$$

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[‡] Our analysis also holds if $f(x) = Ce^x$ where C is a constant.

So $f(1.2) \cong 1.1e^{1.1}$. Or

$$e^{1.2} \cong (1.1)^2 e \tag{6}$$

Eventually we will get e^2 on the left side of the equation, so we can solve for e. So let's compute $e^{1.3}$. We get $e^{1.3} \cong 1.1e^{1.2}$ But this equals $(1.1)^3 e$. If we extrapolate to $x = 1.8$, we see that

$$e^{1.9} \cong (1.1)^9 e \tag{7}$$

and finally that

$$e^2 \cong (1.1)^{10} e \tag{8}$$

Solving for e we get $e \cong (1.1)^{10}$ or e equals 2.59 to three digits, where the 10 corresponds to dividing 1 by 0.1. Equation (1) presupposes that Δx approaches zero. If we let $\Delta x = .00000001$, or 10^{-8}, we raise $(1 + .00000001)$ to 10^8. The answer for e is 2.71828 to five significan figures.

2. Generalization.

We now sketch the steps that describe the preceding method in general. Using equation (2), and setting $x = 1$, we write

$$e^{1+\Delta x} \cong e(1 + \Delta x) \tag{9}$$

We continue, letting $x = x + \Delta x$ and keeping Δx the same, and write

$$e^{1+\Delta x+\Delta x} \cong e^{1+\Delta x}(1 + \Delta x) \tag{10}$$

or

$$e^{1+2\Delta x} \cong e(1 + \Delta x)^2 \tag{11}$$

We have to add Δx to x $1/\Delta x$ times to get e^2 on the left side of these equations. So we get

$$e^{1+(1/\Delta x)\cdot\Delta x} \cong e(1 + \Delta x)^{1/\Delta x} \tag{12}$$

or

$$e^2 = e(1 + \Delta x)^{1/\Delta x} \tag{13}$$

Solve for e and since the definition of the derivative in equation (1) lets $\Delta x \to 0$, take the same limit here. We get

$$e = \lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x} \tag{14}$$

which is one of the definitions of e.