VAR Models and Cointegration [1]

The Cointegrated VAR

In a series of important papers Soren Johansen firmly roots cointegration and error correction models in a vector autoregression framework.

Consider the levels VAR(p) for the ($n \times 1$) vector $\boldsymbol{Y}_t$

$$\boldsymbol{Y}_t = \boldsymbol{\Phi}\boldsymbol{D}_t + \boldsymbol{\Pi}_1\boldsymbol{Y}_{t-1} + \cdots + \boldsymbol{\Pi}_p\boldsymbol{Y}_{t-p} + \boldsymbol{\varepsilon}_t, \quad t = 1, \ldots, T$$

$$\boldsymbol{D}_t = \text{deterministic terms}$$

- The VAR(p) model is stable if

$$\det(\boldsymbol{I}_n - \boldsymbol{\Pi}_1 z - \cdots - \boldsymbol{\Pi}_p z^p) = 0$$

  has all roots outside the complex unit circle.
- If there are roots on the unit circle then some or all of the variables in $\boldsymbol{Y}_t$ are $I(1)$ and they may also be cointegrated.
- If Y_t is cointegrated then the VAR representation is not the most suitable representation for analysis because the cointegrating relations are not explicitly apparent.

- The cointegrating relations become apparent if the levels VAR is transformed to the *vector error correction model* (VECM)

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Phi} \boldsymbol{D}_t + \boldsymbol{\Pi} \boldsymbol{Y}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{Y}_{t-1} + \ldots + \boldsymbol{\Gamma}_{p-1} \Delta \boldsymbol{Y}_{t-p+1} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_1 + \ldots + \boldsymbol{\Pi}_p - \boldsymbol{I}_n$$

$$\boldsymbol{\Gamma}_k = - \sum_{j=k+1}^{p} \boldsymbol{\Pi}_j, \ k = 1, \ldots, p-1$$

- In the VECM, $\Delta \boldsymbol{Y}_t$ and its lags are $I(0)$.

- The term $\boldsymbol{\Pi} \boldsymbol{Y}_{t-1}$ is the only one which includes potential $I(1)$ variables and for $\Delta \boldsymbol{Y}_t$ to be $I(0)$ it must be the case that $\boldsymbol{\Pi} \boldsymbol{Y}_{t-1}$ is also $I(0)$. Therefore, $\boldsymbol{\Pi} \boldsymbol{Y}_{t-1}$ must contain the cointegrating relations if they exist.

- If the VAR(p) process has unit roots ($z = 1$) then

$$\det(\boldsymbol{I}_n - \boldsymbol{\Pi}_1 - \ldots - \boldsymbol{\Pi}_p) = 0 \quad \Rightarrow \det(\boldsymbol{\Pi}) = 0 \quad \Rightarrow \boldsymbol{\Pi} \text{ is singular}$$

If $\boldsymbol{\Pi}$ is singular then it has *reduced rank*; that is $rank(\boldsymbol{\Pi}) = r < n$. There are two cases to consider:

1. $rank(\boldsymbol{\Pi}) = 0$. This implies that

$$\boldsymbol{\Pi} = \mathbf{0}$$

$$\boldsymbol{Y}_t \sim I(1) \text{ and not cointegrated}$$

The VECM reduces to a VAR$(p-1)$ in first differences

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Phi} \boldsymbol{D}_t + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{Y}_{t-1} + \ldots + \boldsymbol{\Gamma}_{p-1} \Delta \boldsymbol{Y}_{t-p+1} + \boldsymbol{\varepsilon}_t$$

2. $0 < rank(\boldsymbol{\Pi}) = r < n$. This implies that $\boldsymbol{Y}_t$ is $I(1)$ with r linearly independent cointegrating vectors and $n - r$ common stochastic trends (unit roots). Since $\boldsymbol{\Pi}$ has rank r it can be written as the product

$$\underset{(n \times n)}{\boldsymbol{\Pi}} = \underset{(n \times r)}{\boldsymbol{\alpha}} \underset{(r \times n)}{\boldsymbol{\beta}'}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $(n \times r)$ matrices with $rank(\boldsymbol{\alpha}) = rank(\boldsymbol{\beta}) = r$. The rows of $\boldsymbol{\beta}'$ form a basis for the r cointegrating vectors and the elements of $\boldsymbol{\alpha}$ distribute the impact of the cointegrating vectors to the evolution of $\Delta \boldsymbol{Y}_t$. The VECM becomes

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Phi} \boldsymbol{D}_t + \boldsymbol{\alpha}\boldsymbol{\beta}' \boldsymbol{Y}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{Y}_{t-1} + \ldots + \boldsymbol{\Gamma}_{p-1}\Delta \boldsymbol{Y}_{t-p+1} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\beta}' \boldsymbol{Y}_{t-1} \sim I(0)$ since $\boldsymbol{\beta}'$ is a matrix of cointegrating vectors.

Non-uniqueness

It is important to recognize that the factorization $\Pi = \alpha\beta'$ is not unique since for any $r \times r$ nonsingular matrix H we have

$$\alpha\beta' = \alpha H H^{-1} \beta' = (\alpha H)(\beta H^{-1\prime})' = \alpha^* \beta^{*\prime}$$

Hence the factorization $\Pi = \alpha\beta'$ only identifies the space spanned by the cointegrating relations. To obtain unique values of α and β' requires further restrictions on the model.

Example: A bivariate cointegrated VAR(1) model

Consider the bivariate VAR(1) model for $\boldsymbol{Y}_t = (y_{1t}, y_{2t})'$

$$\boldsymbol{Y}_t = \boldsymbol{\Pi}_1 \boldsymbol{Y}_{t-1} + \boldsymbol{\epsilon}_t$$

The VECM is

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Pi} \boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_1 - \boldsymbol{I}_2$$

Assuming $\boldsymbol{Y}_t$ is cointegrated there exists a 2×1 vector $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ such that

$$\boldsymbol{\beta}' \boldsymbol{Y}_t = \beta_1 y_{1t} + \beta_2 y_{2t} \sim I(0)$$

Using the normalization $\beta_1 = 1$ and $\beta_2 = -\beta$ the cointegrating relation becomes

$$\boldsymbol{\beta}' \boldsymbol{Y}_t = y_{1t} - \beta y_{2t}$$

This normalization suggests the stochastic long-run equilibrium relation

$$y_{1t} = \beta y_{2t} + u_t$$

Since $\boldsymbol{Y}_t$ is cointegrated with one cointegrating vector, $rank(\boldsymbol{\Pi}) = 1$ so that

$$\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}' = \left(\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) \left(\begin{array}{cc} 1 & -\beta \end{array} \right) = \left(\begin{array}{cc} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{array} \right)$$

The elements in the vector $\boldsymbol{\alpha}$ are interpreted as *speed of adjustment* coefficients. The cointegrated VECM for $\Delta\boldsymbol{Y}_t$ may be rewritten as

$$\Delta\boldsymbol{Y}_t = \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t$$

Writing the VECM equation by equation gives

$$\begin{array}{rcl} \Delta y_{1t} & = & \alpha_1(y_{1t-1} - \beta y_{2t-1}) + \varepsilon_{1t} \\ \Delta y_{2t} & = & \alpha_2(y_{1t-1} - \beta y_{2t-1}) + \varepsilon_{2t} \end{array}$$

The stability conditions for the bivariate VECM are related to the stability conditions for the disequilibrium error $\boldsymbol{\beta}'\boldsymbol{Y}_t$.

It is straightforward to show that $\boldsymbol{\beta}' \boldsymbol{Y}_t$ follows an AR(1) process

$$\boldsymbol{\beta}' \boldsymbol{Y}_t = (1 + \boldsymbol{\beta}' \boldsymbol{\alpha}) \boldsymbol{\beta}' \boldsymbol{Y}_{t-1} + \boldsymbol{\beta}' \boldsymbol{\varepsilon}_t$$

or

$$
\begin{aligned}
u_t &= \phi u_{t-1} + v_t, \ u_t = \boldsymbol{\beta}' \boldsymbol{Y}_t \\
\phi &= 1 + \boldsymbol{\beta}' \boldsymbol{\alpha} = 1 + (\alpha_1 - \beta \alpha_2) \\
v_t &= \boldsymbol{\beta}' \boldsymbol{\varepsilon}_t = u_{1t} - \beta u_{2t}
\end{aligned}
$$

The AR(1) model for u_t is stable as long as

$$|\phi| = |1 + (\alpha_1 - \beta \alpha_2)| < 1$$

For example, suppose $\beta = 1$. Then the stability condition is

$$|\phi| = |1 + (\alpha_1 - \alpha_2)| < 1$$

which is satisfied if

$$\alpha_1 - \alpha_2 < 0 \text{ and } \alpha_1 - \alpha_2 > -2.$$

Johansen's Methodology for Modeling Cointegration

The basic steps in Johansen's methodology are:

- ▶ Specify and estimate a VAR(p) model for $\boldsymbol{Y}_t$.
- ▶ Construct likelihood ratio tests for the rank of $\boldsymbol{\Pi}$ to determine the number of cointegrating vectors.
- ▶ If necessary, impose normalization and identifying restrictions on the cointegrating vectors.
- ▶ Given the normalized cointegrating vectors estimate the resulting cointegrated VECM by maximum likelihood.

LR Tests for the Number of Cointegrating Vectors

The unrestricted cointegrated VECM is denoted $H(r)$. The $I(1)$ model $H(r)$ can be formulated as the condition that the rank of Π is less than or equal to r. This creates a nested set of models

$$
\begin{aligned}
H(0) &\subset \cdots \subset H(r) \subset \cdots \subset H(n) \\
H(0) &= \text{non-cointegrated VAR} \\
H(n) &= \text{stationary VAR(p)}
\end{aligned}
$$

This nested formulation is convenient for developing a sequential procedure to test for the number r of cointegrating relationships.

Remarks:

- ▶ Since the rank of the long-run impact matrix $\boldsymbol{\Pi}$ gives the number of cointegrating relationships in $\boldsymbol{Y}_t$, Johansen formulates likelihood ratio (LR) statistics for the number of cointegrating relationships as LR statistics for determining the rank of $\boldsymbol{\Pi}$.

- ▶ These LR tests are based on the estimated eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \cdots > \hat{\lambda}_n$ of the matrix $\boldsymbol{\Pi}$. These eigenvalues also happen to equal the squared *canonical correlations* between $\Delta \boldsymbol{Y}_t$ and $\boldsymbol{Y}_{t-1}$ corrected for lagged $\Delta \boldsymbol{Y}_t$ and $\boldsymbol{D}_t$ and so lie between 0 and 1. Recall, the rank of $\boldsymbol{\Pi}$ is equal to the number of non-zero eigenvalues of $\boldsymbol{\Pi}$.

Johansen's Trace Statistic

Johansen's LR statistic tests the nested hypotheses

$$H_0(r): r = r_0 \text{ vs. } H_1(r_0): r > r_0$$

The LR statistic, called the *trace statistic*, is given by

$$LR_{trace}(r_0) = -T \sum_{i=r_0+1}^{n} \ln(1 - \hat{\lambda}_i)$$

- If $rank(\Pi) = r_0$ then $\hat{\lambda}_{r_0+1}, \ldots, \hat{\lambda}_n$ should all be close to zero and $LR_{trace}(r_0)$ should be small since $\ln(1 - \hat{\lambda}_i) \approx 0$ for $i > r_0$.

- In contrast, if $rank(\Pi) > r_0$ then some of $\hat{\lambda}_{r_0+1}, \ldots, \hat{\lambda}_n$ will be nonzero (but less than 1) and $LR_{trace}(r_0)$ should be large since $\ln(1 - \hat{\lambda}_i) << 0$ for some $i > r_0$.

▶ The asymptotic null distribution of $LR_{trace}(r_0)$ is not chi-square but instead is a multivariate version of the Dickey-Fuller unit root distribution which depends on the dimension $n - r_0$ and the specification of the deterministic terms. Critical values for this distribution are tabulated in Osterwald-Lenum (1992) for $n - r_0 = 1, \ldots, 10$.

Determining the Number of Cointegrating Vectors

A sequential procedure is used to determine the number of cointegrating vectors:

1. First test $H_0(r_0 = 0)$ against $H_1(r_0 > 0)$. If this null is not rejected then it is concluded that there are no cointegrating vectors among the n variables in $\boldsymbol{Y}_t$.

2. If $H_0(r_0 = 0)$ is rejected then it is concluded that there is at least one cointegrating vector and proceed to test $H_0(r_0 = 1)$ against $H_1(r_0 > 1)$. If this null is not rejected then it is concluded that there is only one cointegrating vector.

3. If the $H_0(r_0 = 1)$ is rejected then it is concluded that there is at least two cointegrating vectors.

4. The sequential procedure is continued until the null is not rejected.

Johansen's Maximum Eigenvalue Statistic

Johansen also derives a LR statistic for the hypotheses

$$H_0(r_0) : r = r_0 \text{ vs. } H_1(r_0) : r_0 = r_0 + 1$$

The LR statistic, called the maximum eigenvalue statistic, is given by

$$LR_{\max}(r_0) = -T \ln(1 - \hat{\lambda}_{r_0+1})$$

As with the trace statistic, the asymptotic null distribution of $LR_{\max}(r_0)$ is not chi-square but instead is a complicated function of Brownian motion, which depends on the dimension $n - r_0$ and the specification of the deterministic terms. Critical values for this distribution are tabulated in Osterwald-Lenum (1992) for $n - r_0 = 1, \ldots, 10$.

Specification of Deterministic Terms

Following Johansen (1995), the deterministic terms in are restricted to the form

$$\mathbf{\Phi} \boldsymbol{D}_t = \boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$$

If the deterministic terms are unrestricted then the time series in $\boldsymbol{Y}_t$ may exhibit quadratic trends and there may be a linear trend term in the cointegrating relationships. Restricted versions of the trend parameters $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ limit the trending nature of the series in $\boldsymbol{Y}_t$. The trend behavior of $\boldsymbol{Y}_t$ can be classified into five cases:

1. Model $H_2(r)$: $\mu_t = 0$ (no constant):

$$\Delta Y_t = \alpha \beta' Y_{t-1} + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \varepsilon_t$$

   and all the series in Y_t are $I(1)$ without drift and the cointegrating relations $\beta' Y_t$ have mean zero.

2. Model $H_1^*(r)$: $\mu_t = \mu_0 = \alpha \rho_0$ (restricted constant):

$$\Delta Y_t = \alpha(\beta' Y_{t-1} + \rho_0) + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \varepsilon_t$$

   the series in Y_t are $I(1)$ without drift and the cointegrating relations $\beta' Y_t$ have non-zero means ρ_0.

3. Model $H_1(r)$: $\mu_t = \mu_0$ (unrestricted constant):

$$\Delta Y_t = \mu_0 + \alpha \beta' Y_{t-1} + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \varepsilon_t$$

   the series in Y_t are $I(1)$ with drift vector μ_0 and the cointegrating relations $\beta' Y_t$ may have a non-zero mean.

4. Model $H^*(r)$: $\mu_t = \mu_0 + \alpha\rho_1 t$ (restricted trend). The restricted VECM is

$$
\begin{aligned}
\Delta Y_t &= \mu_0 + \alpha(\beta' Y_{t-1} + \rho_1 t) \\
&\quad + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \varepsilon_t
\end{aligned}
$$

the series in Y_t are $I(1)$ with drift vector μ_0 and the cointegrating relations $\beta' Y_t$ have a linear trend term $\rho_1 t$.

5. Model $H(r)$: $\mu_t = \mu_0 + \mu_1 t$ (unrestricted constant and trend). The unrestricted VECM is

$$
\begin{aligned}
\Delta Y_t &= \mu_0 + \mu_1 t + \alpha\beta' Y_{t-1} \\
&\quad + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \varepsilon_t
\end{aligned}
$$

the series in Y_t are $I(1)$ with a linear trend (quadratic trend in levels) and the cointegrating relations $\beta' Y_t$ have a linear trend.

Case 1: No constant
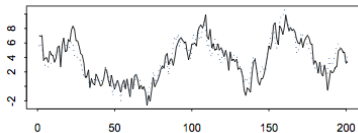
Case 2: Restricted constant
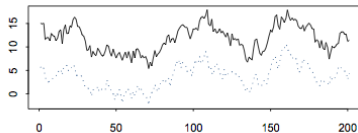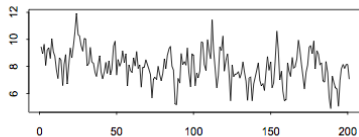
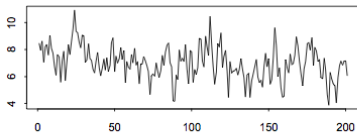Case 3: Unrestricted constant
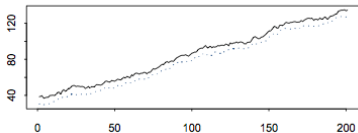
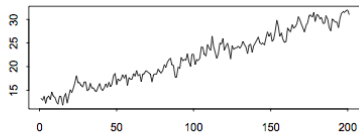Case 4: Restricted trend
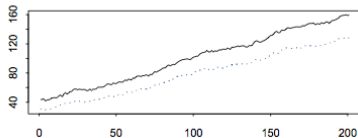
Case 5: Unrestricted trend
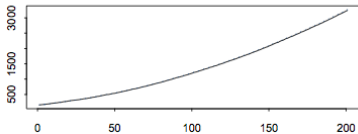
Case 1: No constant
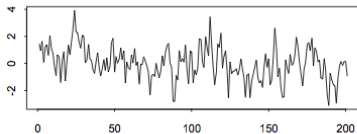
Case 2: Restricted constant

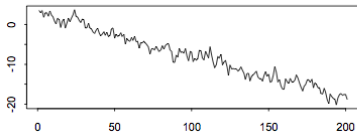Case 3: Unrestricted constant

Case 4: Restricted trend

Case 5: Unrestricted trend

Johansen's Cointegration Tests in R

The package *urca* includes functions for cointegration analysis based on Johansen's methods.

```
> ca.jo
function (x, type = c("eigen", "trace"), ecdet = c("none", "const",
    "trend"), K = 2, spec = c("longrun", "transitory"), season = NULL,
    dumvar = NULL)
```

Note:

- ▶ This function considers 3 specifications for the deterministic term:
  - ▶ "none" corresponds to case 1, i.e. no constant
  - ▶ "const" corresponds to case 2, i.e. restricted constant
  - ▶ "trend" corresponds to case 4, i.e. restricted trend
- ▶ K is the number of lags for the VAR in levels so $K - 1$ is the number of lags in the VECM representation

Example 1: Testing for Cointegration

Simulated data set 1:

$$
\begin{aligned}
y_{t1} &= y_{2t} + u_t, \quad u_t = 0.75 u_{t-1} + \varepsilon_{1t} \\
y_{2t} &= y_{2t-1} + \varepsilon_{2t} \\
\varepsilon_{it} &\sim iid\ N(0, (0.5)^2) \text{ for } i = 1, 2
\end{aligned}
$$

```
> # Example bivariate cointegrated system
> e1 <- rnorm(250,mean=0,sd=.5)
> e2 <- rnorm(250,mean=0,sd=.5)
> u.ar1 <- arima.sim(model=list(ar=.75),250,innov=e1)
> y2 <- cumsum(e2)
> y1 <- y2 + u.ar1
> data1 <- cbind(y1,y2)
```

Example 1: Trace Statistic

```
> # trace statistic
> test1 <- ca.jo(data1,ecdet="const",type="trace",K=2,spec="transitory")
> summary(test1)


Test type: trace statistic , without linear trend and constant in cointeg


Eigenvalues (lambda):
[1] 9.979511e-02 1.840829e-02 4.582038e-18


Values of teststatistic and critical values of test:


          test 10pct  5pct  1pct
r <= 1 |  4.61  7.52  9.24 12.97
r = 0  | 30.68 17.85 19.96 24.60
```

▶ Test finds 1 cointegrating vector

Example 1: Max Eigenvalue Statistic

```
> # max eigenvalue statistic
> test2 <- ca.jo(data1,ecdet="const",type="eigen",K=2,spec="transitory")
> summary(test2)


Test type: maximal eigenvalue statistic (lambda max) , without linear tre

Eigenvalues (lambda):
[1] 9.979511e-02 1.840829e-02 4.582038e-18


Values of teststatistic and critical values of test:


          test 10pct  5pct  1pct
r <= 1 |  4.61  7.52  9.24 12.97
r = 0  | 26.07 13.75 15.67 20.20
```

- ▶ Test finds 1 cointegrating vector

Example 2: Testing for Cointegration

Simulated data set 2:

$$
\begin{aligned}
y_{t1} &= 0.5y_{2t} + 0.5y_{3t} + u_t, \quad u_t = 0.75u_{t-1} + \varepsilon_{1t} \\
y_{2t} &= y_{2t-1} + \varepsilon_{2t} \\
y_{3t} &= y_{3t-1} + \varepsilon_{3t} \\
\varepsilon_{it} &\sim iid\ N(0, (0.5)^2) \text{ for } i = 1, 2, 3
\end{aligned}
$$

```
> # Example trivariate cointegrated system (1 coint vector)
> e1 <- rnorm(250,mean=0,sd=.5)
> e2 <- rnorm(250,mean=0,sd=.5)
> e3 <- rnorm(250,mean=0,sd=.5)
> u.ar1 <- arima.sim(model=list(ar=.75),250,innov=e1)
> y2 <- cumsum(e2)
> y3 <- cumsum(e3)
> y1 <- .5*y2 + .5*y3 + u.ar1
> data2 <- cbind(y1,y2,y3)
```

Example 2: Trace Statistic

```
> # trace statistic
> test1 <- ca.jo(data2,ecdet="const",type="trace",K=2,spec="transitory")
> summary(test1)


Test type: trace statistic , without linear trend and constant in cointe


Eigenvalues (lambda):
[1] 1.193654e-01 3.702304e-02 1.051500e-02 3.592029e-18


Values of teststatistic and critical values of test:

          test 10pct  5pct  1pct
r <= 2 |  2.62  7.52  9.24 12.97
r <= 1 | 11.98 17.85 19.96 24.60
r = 0  | 43.50 32.00 34.91 41.07
```

- ▶ Test finds 1 cointegrating vector

Example 3: Testing for Cointegration

Simulated data set 3:

$$y_{1t} = y_{3t} + u_t, \quad u_t = 0.75u_{t-1} + \varepsilon_{1t}$$

$$y_{2t} = y_{3t} + v_t, \quad v_t = 0.75v_{t-1} + \varepsilon_{2t}$$

$$y_{3t} = y_{3t-1} + \varepsilon_{3t}$$

$$\varepsilon_{it} \sim iid\ N(0, (0.5)^2) \text{ for } i = 1, 2, 3$$

```
> # Example trivariate cointegrated system (2 coint vectors)
> e1 <- rnorm(250,mean=0,sd=.5)
> e2 <- rnorm(250,mean=0,sd=.5)
> e3 <- rnorm(250,mean=0,sd=.5)
> u.ar1 <- arima.sim(model=list(ar=.75),250,innov=e1)
> v.ar1 <- arima.sim(model=list(ar=.75),250,innov=e2)
> y3 <- cumsum(e3)
> y1 <- y3 + u.ar1
> y2 <- y3 + v.ar1
> data3 <- cbind(y1,y2,y3)
```

Example 3: Trace Statistic

```
> # trace statistic
> test1 <- ca.jo(data3,ecdet="const",type="trace",K=2,spec="transitory")
> summary(test1)

Test type: trace statistic , without linear trend and constant in cointe

Eigenvalues (lambda):
[1]  1.303347e-01  9.747977e-02  9.712308e-03 -2.544989e-18

Values of teststatistic and critical values of test:

          test 10pct  5pct  1pct
r <= 2 |  2.42  7.52  9.24 12.97
r <= 1 | 27.86 17.85 19.96 24.60
r = 0  | 62.49 32.00 34.91 41.07
```

▶ Test finds 2 cointegrating vectors

Maximum Likelihood Estimation of the VECM

If it is found that $rank(\mathbf{\Pi}) = r$, $0 < r < n$, then the cointegrated VECM

$$\Delta \mathbf{Y}_t = \mathbf{\Phi} \mathbf{D}_t + \mathbf{\alpha}\mathbf{\beta}'\mathbf{Y}_{t-1} + \mathbf{\Gamma}_1\Delta \mathbf{Y}_{t-1} + \ldots + \mathbf{\Gamma}_{p-1}\Delta \mathbf{Y}_{t-p+1} + \mathbf{\varepsilon}_t$$

becomes a reduced rank multivariate regression. Johansen derived the maximum likelihood estimation of the parametes under the reduced rank restriction $rank(\mathbf{\Pi}) = r$ (see Hamilton for details). He shows that

- $\hat{\mathbf{\beta}}_{mle} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_r)$, where $\hat{\mathbf{v}}_i$ are the eigenvectors associated with the eigenvalues $\hat{\lambda}_i$
- The MLEs of the remaining parameters are obtained by least squares estimation of

$$\Delta \mathbf{Y}_t = \mathbf{\Phi} \mathbf{D}_t + \mathbf{\alpha}\hat{\mathbf{\beta}}'_{mle}\mathbf{Y}_{t-1} + \mathbf{\Gamma}_1\Delta \mathbf{Y}_{t-1} + \ldots + \mathbf{\Gamma}_{p-1}\Delta \mathbf{Y}_{t-p+1} + \mathbf{\varepsilon}_t$$

Normalized Estimates of α and β

- The factorization

$$\hat{\Pi}_{mle} = \hat{\alpha}_{mle} \; \hat{\beta}'_{mle}$$

  is not unique.

- The columns of $\hat{\beta}_{mle}$ may be interpreted as linear combinations of the underlying cointegrating relations.

- For interpretations, it is often convenient to normalize or identify the cointegrating vectors by choosing a specific coordinate system in which to express the variables.

Johansen's normalized MLE

- An arbitrary normalization, suggested by Johansen, is to solve for the triangular representation of the cointegrated system. The resulting normalized cointegrating vector is denoted $\hat{\boldsymbol{\beta}}_{c,mle}$. The normalization of the MLE for $\boldsymbol{\beta}$ to $\hat{\boldsymbol{\beta}}_{c,mle}$ will affect the MLE of $\boldsymbol{\alpha}$ but not the MLEs of the other parameters in the VECM.

- Let $\hat{\boldsymbol{\beta}}_{c,mle}$ denote the MLE of the normalized cointegrating matrix $\boldsymbol{\beta}_c$. Johansen (1995) showed that

$$T(vec(\hat{\boldsymbol{\beta}}_{c,mle}) - vec(\boldsymbol{\beta}_c))$$

is asymptotically (mixed) normally distributed.

- $\hat{\boldsymbol{\beta}}_{c,mle}$ is super consistent.

Testing Linear Restrictions on β

The Johansen MLE procedure only produces an estimate of the basis for the space of cointegrating vectors. It is often of interest to test if some hypothesized cointegrating vector lies in the space spanned by the estimated basis:

$$
H_0 \quad : \quad \underset{(r \times n)}{\beta}{}' = \left(\begin{array}{c} \beta_0' \\ \phi' \end{array} \right)
$$

$$
\beta_0' \quad = \quad s \times n \text{ matrix of hypothesized cv's}
$$

$$
\phi' \quad = \quad (r - s) \times n \text{ matrix of remaining unspecified cv's}
$$

Result: Johansen (1995) showed that a likelihood ratio statistic can be computed, which is asymptotically distributed as a χ^2 with $s(n - r)$ degrees of freedom.

Example 1 continued...

The function *cajorls* (*urca* package) estimates the restricted VECM.

```
> # estimate restricted VECM
> model1 <- cajorls(test1,r=1)
> print(model1)
$rlm


$beta
             ect1
y1.l1     1.00000000
y2.l1    -0.92911069
constant -0.07135761
```

- ▶ Note that the β coefficients closely match those in the DGP

Restricted VECM

```
Response y1.d :

Coefficients:
       Estimate Std. Error t value Pr(>|t|)
ect1   -0.18461    0.05589  -3.303   0.0011 **
y1.dl1  0.16984    0.09057   1.875   0.0619 .
y2.dl1 -0.20509    0.12156  -1.687   0.0928 .

Residual standard error: 0.7182 on 245 degrees of freedom
Multiple R-squared: 0.04769, Adjusted R-squared: 0.03603
F-statistic:  4.09 on 3 and 245 DF,  p-value: 0.007393
```

▶ Note that y_{1t} depends on the cointegrating vector

Restricted VECM

```
Response y2.d :

Coefficients:
       Estimate Std. Error t value Pr(>|t|)
ect1    0.01918    0.04102   0.468    0.640
y1.dl1  0.08472    0.06647   1.274    0.204
y2.dl1 -0.04996    0.08922  -0.560    0.576

Residual standard error: 0.5271 on 245 degrees of freedom
Multiple R-squared: 0.01099, Adjusted R-squared: -0.001121
F-statistic: 0.9075 on 3 and 245 DF,  p-value: 0.438
```

▶ Note that y_{2t} does not depend on the cointegrating vector

Testing Restrictions on β

Suppose we want to test whether the cointegrating vector is $\boldsymbol{\beta}' = (1, -1)$ without imposing any restrictions on the intercept. This hypothesis can be represented as a linear restriction on $\boldsymbol{\beta}$:

$$H_0 : \boldsymbol{R}'\boldsymbol{\beta} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{\beta} = \boldsymbol{B}\boldsymbol{\Psi}$$

where $\boldsymbol{\Psi}$ are the unknown parameters in the cointegrating vectors $\boldsymbol{\beta}$, $\boldsymbol{B} = \boldsymbol{R}_{\perp}$ and $\boldsymbol{R}_{\perp}$ is the orthogonal complement of $\boldsymbol{R}$ such that $\boldsymbol{R}'\boldsymbol{R}_{\perp} = \boldsymbol{0}$.

Testing Restrictions on β

```
> B1 <- matrix(c(1,-1,0,0,0,1),nrow=3)
> B1
     [,1] [,2]
[1,]   1    0
[2,]  -1    0
[3,]   0    1
> R <- Null(B1)
> R
           [,1]
[1,] -0.7071068
[2,] -0.7071068
[3,]  0.0000000
```

▶ Now the test can be simply performed by passing the matrix B to the test function.

Testing Restrictions on β

```
> # test restrictions on beta
> test3 <- blrtest(z=test1,H=B1,r=1)
> summary(test3)

Estimation and testing under linear restrictions on beta

Eigenvalues of restricted VAR (lambda):
[1] 0.0974 0.0000

The value of the likelihood ratio test statistic:
0.65 distributed as chi square with 1 df.
The p-value of the test statistic is: 0.42
```

- ▶ The restrictions are not rejected

Estimate VECM with Restrictions on β

```
> # estimate VECM with beta restrictions
> model2 <- cajorls(test3,r=1)
> print(model2)
$rlm

$beta
                ect1
y1.l1    1.0000000000
y2.l1   -1.0000000000
constant  0.0008041931
```

Testing Restrictions on β

Suppose we want to test whether the cointegrating vector is $\boldsymbol{\beta}' = (1, -1)$ with no intercept.

```
> B2 <- matrix(c(1,-1,0),nrow=3)
> B2
     [,1]
[1,]    1
[2,]   -1
[3,]    0
> R <- Null(B2)
> R
          [,1] [,2]
[1,] 0.7071068    0
[2,] 0.7071068    0
[3,] 0.0000000    1
```

Testing Restrictions on β

```
> # test restrictions on beta
> test4 <- blrtest(z=test1,H=B2,r=1)
> summary(test4)

Estimation and testing under linear restrictions on beta

Eigenvalues of restricted VAR (lambda):
[1] 0.0974

The value of the likelihood ratio test statistic:
0.65 distributed as chi square with 2 df.
The p-value of the test statistic is: 0.72
```

▶ The restrictions are not rejected

Estimate VECM with Restrictions on β

```
> # estimate VECM with beta restrictions
> model3 <- cajorls(test4,r=1)
> print(model3)
$rlm


$beta
        ect1
y1.l1      1
y2.l1     -1
constant   0
```

Testing Restrictions on α

This hypothesis can be represented as a linear restriction on α:

$$H_0 : R'\alpha = 0 \quad \text{or} \quad \alpha = A\Psi$$

```
> A1 <- matrix(c(1,0),nrow=2)
> R <- Null(A1)
> R
     [,1]
[1,]    0
[2,]    1
```

▶ Now the test can be simply performed by passing the matrix A to the test function.

Testing Restrictions on α

```
> # test weak exogeneity
> test5 <- alrtest(z=test1,A=A1,r=1)
> summary(test5)

The value of the likelihood ratio test statistic:
0.18 distributed as chi square with 1 df.
The p-value of the test statistic is: 0.67

Weights W of the restricted VAR:


         [,1]
[1,] -0.2031
[2,]  0.0000
```

- ▶ The restrictions are not rejected

Testing Restrictions on α and β

```
> # test restriction on beta and weak exogeneity
> test6 <- ablrtest(z=test1,A=A1,H=B2,r=1)
> summary(test6)

Estimation and testing under linear restrictions on alpha and beta

Eigenvalues of restricted VAR (lambda):
[1] 0.0949

The value of the likelihood ratio test statistic:
1.34 distributed as chi square with 2 df.
The p-value of the test statistic is: 0.51
```

▶ The restrictions are not rejected

Testing Restrictions on α and β

```
Eigenvectors, normalised to first column
of the restricted VAR:

     [,1]
[1,]    1
[2,]   -1
[3,]    0

Weights W of the restricted VAR:

         [,1]
[1,] -0.1967
[2,]  0.0000

>
```

Example 3 continued...

The function *cajorls* (*urca* package) estimates the restricted VECM.

```
> # estimate restricted VECM
> model1 <- cajorls(test1,r=2)
> print(model1)
$rlm

$beta
                 ect1         ect2
y1.l1     1.000000e+00   0.0000000
y2.l1    -5.551115e-17   1.0000000
y3.l1    -9.730800e-01  -0.9982614
constant -2.526943e-01   0.1276533
```

 ▶ Note that the β coefficients closely match those in the DGP

Restricted VECM

```
Response y1.d :


Coefficients:
      Estimate Std. Error t value Pr(>|t|)
ect1  -0.30095   0.06850  -4.394 1.66e-05 ***
ect2  -0.04163   0.05504  -0.756    0.450
y1.dl1 0.04422   0.09318   0.475    0.636
y2.dl1 -0.02032  0.08383  -0.242    0.809
y3.dl1 -0.03034  0.15012  -0.202    0.840
---


Residual standard error: 0.6869 on 243 degrees of freedom
Multiple R-squared: 0.07999, Adjusted R-squared: 0.06106
F-statistic: 4.226 on 5 and 243 DF,  p-value: 0.001058
```

▶ Note that y_{1t} only depends on the first cointegrating vector

Restricted VECM

```
Response y2.d :

Coefficients:
       Estimate Std. Error t value Pr(>|t|)
ect1   -0.01749    0.07354  -0.238 0.812242
ect2   -0.22410    0.05909  -3.792 0.000188 ***
y1.dl1  0.07135    0.10005   0.713 0.476449
y2.dl1 -0.03753    0.09001  -0.417 0.677095
y3.dl1 -0.11722    0.16118  -0.727 0.467758
---

Residual standard error: 0.7375 on 243 degrees of freedom
Multiple R-squared: 0.07269, Adjusted R-squared: 0.05361
F-statistic: 3.81 on 5 and 243 DF,  p-value: 0.00243
```

▶ Note that y_{2t} only depends on the second cointegrating vector

Restricted VECM

```
Response y3.d :

Coefficients:
       Estimate Std. Error t value Pr(>|t|)
ect1   -0.02667    0.05218  -0.511    0.610
ect2   -0.00841    0.04193  -0.201    0.841
y1.dl1  0.03379    0.07099   0.476    0.634
y2.dl1 -0.02255    0.06386  -0.353    0.724
y3.dl1 -0.05837    0.11436  -0.510    0.610
---

Residual standard error: 0.5233 on 243 degrees of freedom
Multiple R-squared: 0.00485, Adjusted R-squared: -0.01563
F-statistic: 0.2369 on 5 and 243 DF,  p-value: 0.946
```

▶ Note that y_{3t} does not depend on any of the cointegrating vectors

Testing Restrictions on β

```
> # test restrictions on beta
> B1 <- matrix(c(1,0,-1,0,0,1,-1,0,0,0,0,1),nrow=4)
> test3 <- blrtest(z=test1,H=B1,r=2)
> summary(test3)

Estimation and testing under linear restrictions on beta

Eigenvalues of restricted VAR (lambda):
[1] 0.1285 0.0973 0.0006

The value of the likelihood ratio test statistic:
0.56 distributed as chi square with 2 df.
The p-value of the test statistic is: 0.76
```

- ▶ The restrictions are not rejected

Estimate VECM with Restrictions on β

```
> # estimate VECM with beta restrictions
> model2 <- cajorls(test3,r=2)
> print(model2)
$rlm

$beta
             ect1       ect2
y1.l1     1.0000000  0.0000000
y2.l1     0.0000000  1.0000000
y3.l1    -1.0000000 -1.0000000
constant -0.1278010  0.1355248
```

Testing Restrictions on α

```
> # test weak exogeneity
> A1 <- matrix(c(1,0,0,0,1,0),nrow=3)
> test4 <- alrtest(z=test1,A=A1,r=2)
> summary(test4)


The value of the likelihood ratio test statistic:
0.28 distributed as chi square with 2 df.
The p-value of the test statistic is: 0.87


Weights W of the restricted VAR:


         [,1]    [,2]
[1,] -0.2446 -0.0308
[2,] -0.0860  0.0952
[3,]  0.0000  0.0000
```

▶ The restrictions are not rejected

Testing Restrictions on α and β

```
> # test restriction on beta and weak exogeneity
> test5 <- ablrtest(z=test1,A=A1,H=B1,r=2)
> summary(test5)


Estimation and testing under linear restrictions on alpha and beta


Eigenvalues of restricted VAR (lambda):
[1] 0.1281 0.0973 0.0000


The value of the likelihood ratio test statistic:
0.69 distributed as chi square with 2 df.
The p-value of the test statistic is: 0.71
```

▶ The restrictions are not rejected

Zivot (2000) continued...

```
> data <- cbind(st,ft)
>
> # trace test
> test1 <- ca.jo(data,ecdet="const",type="trace",K=2,spec="transitory")
> summary(test1)

Eigenvalues (lambda):
[1] 5.922276e-02 2.205465e-02 3.469447e-18

Values of teststatistic and critical values of test:

          test 10pct  5pct  1pct
r <= 1 |  5.42  7.52  9.24 12.97
r = 0  | 20.25 17.85 19.96 24.60
```

Estimate Restricted VECM

```
> # estimate restricted model
> model1 <- cajorls(test1,r=1)
> print(model1)
$rlm

$beta
                 ect1
st.l1     1.000000000
ft.l1    -1.000860977
constant -0.001915953
```

Restricted VECM

```
Response st.d :

Coefficients:
        Estimate Std. Error t value Pr(>|t|)
ect1    1.65502    0.82642    2.003   0.0463 *
st.dl1  0.03872    1.89308    0.020   0.9837
ft.dl1  0.03062    1.89125    0.016   0.9871
---

Residual standard error: 0.03401 on 240 degrees of freedom
Multiple R-squared: 0.02481, Adjusted R-squared: 0.01262
F-statistic: 2.035 on 3 and 240 DF,  p-value: 0.1096
```

- Note that s_t only depends on the cointegrating vector

Restricted VECM

```
Response ft.d :

Coefficients:
        Estimate Std. Error t value Pr(>|t|)
ect1   1.745423   0.825300   2.115   0.0355 *
st.dl1 0.004534   1.890527   0.002   0.9981
ft.dl1 0.067139   1.888691   0.036   0.9717
---

Residual standard error: 0.03396 on 240 degrees of freedom
Multiple R-squared: 0.02726, Adjusted R-squared: 0.0151
F-statistic: 2.242 on 3 and 240 DF,  p-value: 0.08405
```

▶ Note that f_t only depends on the cointegrating vector

Testing Restrictions on β

```
> # test restrictions on beta
> B1 <- matrix(c(1,-1,0,0,0,1),nrow=3)
> test3 <- blrtest(z=test1,H=B1,r=1)
> summary(test3)

Estimation and testing under linear restrictions on beta

The value of the likelihood ratio test statistic:
0.02 distributed as chi square with 1 df.
The p-value of the test statistic is: 0.88
```

▶ The restrictions are not rejected

Estimate VECM with Restrictions on β

```
> # estimate restricted model
> model2 <- cajorls(test3,r=1)
> print(model2)
$rlm

Coefficients:
        st.d     ft.d
ect1    1.57824  1.67058
st.dl1  0.09174  0.05737
ft.dl1 -0.02217  0.01449


$beta
                 ect1
st.l1     1.000000000
ft.l1    -1.000000000
constant -0.002356319
```