CME 308 Qualifying Exam

Consider the following multi-type population process. There are two types of individuals. At each integer time, each individual of type i ($i = 1, 2$) is independently replaced by $N_{i,1}$ ($N_{i,2}$) individuals of type 1 (type 2). Let

$$p_i(k,l) = \mathrm{P}(N_{i,1} = k, N_{i,2} = l).$$

Suppose that $p_i(k,l) > 0$ for $k, l \in \mathbb{Z}_+$. Let $X_n = (X_n(1), X_n(2))$, where $X_n(i)$ is the number of type-i individuals at time n. Unless otherwise stated, you may express your solutions in terms of the $P((i_1, i_2), (j_1, j_2))$ transition probabilities for X.

a.) Express

$$\mathbb{E}\left[\exp(-\theta_1 X_1(1) - \theta_2 X_1(2)) | X_0 = (i_1, i_2)\right]$$

for $i_1, i_2 \in \mathbb{Z}_+$ and $\theta_1, \theta_2 \in \mathbb{R}_+$ in terms of $p_1(\cdot)$ and $p_2(\cdot)$.

b.) Let $T = \inf\{n \geq 0 : X_n = (0,0)\}$ be the extinction time of the population. Find a linear system of equations satisfied by $u^*(i_1, i_2)$, where

$$u^*(i_1, i_2) = \mathrm{P}(T < \infty | X_0 = (i_1, i_2)).$$

c.) Argue that $u^*(i_1, i_2) = u_1^{i_1} u_2^{i_2}$ for some $u_1, u_2 \in (0, 1]$.

d.) Find a (non-linear) system of equations satisfied by u_1 and u_2, expressed in terms of $p_1(\cdot)$ and $p_2(\cdot)$.

e.) Argue that X, conditional on $T < \infty$, is Markov and compute its one-step transition probabilities.

f.) Suppose that $N_{i,1}$ and $N_{i,2}$ are independent Poisson rv's, with mean $\lambda^* p$ and $\lambda^* r$, respectively, where p and r are known. Assume that we observe $X_0(1)$ and X_1. What is the MLE for λ^*? (Hint: Recall that the sum of a Poisson rv with mean λ_1 and an independent Poisson rv with mean λ_2 is a Poisson rv with mean $\lambda_1 + \lambda_2$.)

g.) Suppose that $\mathrm{P}(T < \infty | X_0 = (i_1, i_2))$ for $(i_1, i_2) \in \mathbb{Z}_+^2$. Write down the optimality equation for the problem of maximizing

$$\max_{\tau} \mathbb{E}[(aX_\tau(1) + b\tau) I(\tau < T) | X_0 = (i_1, i_2)]$$

for $a, b > 0$.