

ICME Qualifying Exam, June 2009
Discrete Mathematics and Algorithms

1. Let $G(V, E)$ be an undirected graph with n nodes and m edges. For a subset $X \subseteq V$, we use $G[X]$ to denote the subgraph *induced* on X - that is, the graph whose node set is X and whose edge set consists of all edges of G for which both ends lie in X .

Give a polynomial-time algorithm that produces, for a given natural number $k \leq n$, a set $X \subseteq V$ of k nodes with the property that the induced subgraph $G[X]$ has at least $\frac{mk(k-1)}{n(n-1)}$ edges.

You may give either a deterministic algorithm, or a randomized algorithm that has an expected running time that is polynomial, and that only outputs correct answers.

2. A d -coloring of a graph $G(V, E)$ is an assignment of one of d possible colors to each vertex of G such that no two adjacent vertices receive the same color. Prove the following:
 - a. If the maximum degree in G is d , then G is $d + 1$ -colorable.
 - b. If the maximum degree in G is d and G is connected, then G is d -colorable unless it is a complete graph or an odd cycle.