Homework 5
Due 2006/3/26

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1. Prove or disprove:

   - $\mathrm{Tr}(|A+B|) \leq \mathrm{Tr}(|A|) + \mathrm{Tr}(|B|)$.
   - For A, B Hermitian, $\sqrt{A \otimes B} = \sqrt{A} \otimes \sqrt{B}$.
   - $A \geq B$, $C \geq D$ imply $AC \geq BD$.
   - If $A \geq 0$ then so is: (i) A^{-1} (if exists), (ii) $C^\dagger AC$ for any operator C.

2. Find Schmidt decompositions for:

   - $\frac{1}{2}(|00\rangle + |10\rangle - |10\rangle + |11\rangle)$,
   - $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

3. We now prove that the trace distance, the square root of the fidelity and the fidelity itself are convex/concave. Let $p_i \geq 0$ such that $\sum_i p_i = 1$. Prove that:

   - $\|\rho - \sum p_i \sigma_i\|_{tr} \leq \sum_i p_i \|\rho - \sigma_i\|_{tr}$.
   - $\sqrt{F}(\rho, \sum_i p_i \sigma_i) \geq \sum_i p_i \sqrt{F}(\rho, \sigma_i)$.
   - Prove that if the classical fidelity F (over probability distributions) is concave, then the quantum fidelity (over density matrices) is concave.
   - * Prove that $F(\rho, \sum_i p_i \sigma_i) \geq \sum_i p_i F(\rho, \sigma_i)$ for the *classical* fidelity function F.

4. For every $\delta \in [0,2]$ find two pairs of probability distributions (p_1, q_1) and (p_2, q_2) such that $|p_1 - q_1|_1 = |p_2 - q_2|_1 = \delta$ but in the first case the fidelity matches the lower bound $\sqrt{F}(p_1, q_1) = 1 - \frac{\delta}{2}$ whereas in the other it matches the upper bound $F(p_2, q_2) = 1 - \delta^2/4$.