Homework 2
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Instructions as before.

1. **Dictatorship test with perfect completeness:** Prove that there is no 3-query dictatorship test that uses only tests of the form $f(x)f(y)f(z) = 1$ and $f(x)f(y)f(z) = -1$ and has perfect completeness (i.e., accepts dictatorships with probability 1) and reject functions ε-far from dictatorships with some nonzero probability.

2. **Testing resiliency:** We call a function $f : \{0,1\}^n \to \{-1,1\}$ 1-*resilient* if $\widehat{f}(S) = 0$ for all $|S| \le 1$.
   (a) Give a combinatorial definition of 1-resiliency.
   (b) Give a poly$(1/\varepsilon)$-query test that accepts 1-resilient functions with probability at least 2/3, and rejects functions with $|\widehat{f}(S)| \ge \varepsilon$ for some $|S| \le 1$ with probability at least 2/3. (Notice that this is not quite the same thing as a tester for the property of being 1-resilient.) Do this by amplifying a 2-query test.

3. **Tribes function:** For any k, l we define the *tribes function* $f : \{0,1\}^n \to \{-1,1\}$ on $n = kl$ variables as

$$f(x_1, \ldots, x_n) = OR(AND(x_1, \ldots, x_l), AND(x_{l+1}, \ldots, x_{2l}), \ldots, AND(x_{(k-1)l+1}, \ldots, x_{kl})).$$

   (a) Compute the influence of each of its variables.
   (b) Show that for any l, there is a way to choose k such that the tribes function is more-or-less balanced (or more precisely, that the limit of Exp$[f]$ is 0 as l goes to infinity).
   (c) Compare the maximum influence of the balanced tribes function with that of the majority function.

4. **Quasirandomness implies low correlation with juntas:**
   (a) For $f, g : \{0,1\}^n \to \mathbb{R}$ define $\mathrm{Cov}[f,g] := \mathrm{Exp}_x[f(x)g(x)] - \mathrm{Exp}_x[f(x)]\,\mathrm{Exp}_x[g(x)]$. Find an expression for $\mathrm{Cov}[f,g]$ in term of the Fourier coefficients of f and g.
   (b) Show that for any (ε, δ)-quasirandom function $h : \{0,1\}^n \to [-1,1]$ and any r-junta $f : \{0,1\}^n \to \{-1,1\}$, $\mathrm{Cov}[h,f] < \sqrt{\varepsilon r/(1-\delta)^r}$. Notice that this result is trivial for $r \ge \ln(1/\varepsilon)/\delta$. Hint: recall the Cauchy-Schwarz inequality $\sum a_i b_i \le \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.

5. **Compactly storing a function:** Let $f : \{0,1\}^n \to \mathbb{R}$ be some function, and assume we want to store some information about f that would allow us to compute $f(x)$ for any given $x \in \{0,1\}^n$ to within some accuracy, say, ± 0.01. Without any further restrictions on f we would have to store $\Omega(2^n)$ bits of information (even for a Boolean f).
   (a) Show how to reduce the storage to poly(n) for functions f with the property that for all S, $\widehat{f}(S) \ge 0$ (such functions are called *positive definite*) and moreover, $\sum_S \widehat{f}(S) = 1$. Notice that if $f = g \star g$ for some Boolean g then it satisfies these two requirements.
   (b) Extend this to functions f satisfying $\sum_S |\widehat{f}(S)| \le$ poly(n).

6. **Enflo's distortion lower bound on embedding ℓ_1 into ℓ_2 [2]:** The hypercube $\{0,1\}^n$ with the Hamming distance is an ℓ_1 *metric space* (because we can map $\{0,1\}^n$ to $\mathbb{R}^n$ in such a way that the Hamming distance is mapped exactly to the ℓ_1 distance). We say that the hypercube can be *embedded into ℓ_2 with distortion D* if there exists a mapping $F : \{0,1\}^n \to \mathbb{R}^m$ for some m such that for all $x, y \in \{0,1\}^n$,

$$\Delta(x,y) \leq \|F(x) - F(y)\|_2 \leq D \cdot \Delta(x,y).$$

This means that the ℓ_2 distance between $F(x)$ and $F(y)$ is the same as the Hamming distance between x and y up to a factor of D. It is easy to see that there exists an embedding with distortion $\sqrt{n}$. Here we show that this is optimal, and hence this gives an example of an ℓ_1 metric with N points whose distortion when embedded into ℓ_2 is $\sqrt{\log N}$. It was recently shown that any ℓ_1 metric with N points can be embedded into ℓ_2 with distortion $O(\sqrt{\log N} \log\log N)$ [1] (see also [3]).

   (a) Show that for any $f : \{0,1\}^n \to \mathbb{R}$,

$$\operatorname*{Exp}_x[(f(x) - f(x \oplus (1,\ldots,1)))^2] = 4\|f^{odd}\|_2^2 \leq 4\,\mathrm{Var}[f] \leq 4\mathbb{I}(f) = \sum_{i=1}^{n} \operatorname*{Exp}_x[(f(x) - f(x \oplus e_i))^2].$$

   (b) Deduce that for any $F : \{0,1\}^n \to \mathbb{R}^m$,

$$\operatorname*{Exp}_x[\|F(x) - F(x \oplus (1,\ldots,1))\|_2^2] \leq \sum_{i=1}^{n} \operatorname*{Exp}_x[\|F(x) - F(x \oplus e_i)\|_2^2].$$

   (c) Use this to conclude that the distortion of any $F : \{0,1\}^n \to \mathbb{R}^m$ must be at least $\sqrt{n}$.

7. **A hardness reduction that fails:** Consider the following attempt to show that for any $\eta > 0$, $(\frac{1}{2} + \eta, 1 - \eta)$-MAX3LIN is unique-games-hard. Given a unique CSP $G = (V, E)$ over alphabet $[k]$ we reduce it to the following tester over $2^k \cdot |V|$ Boolean variables representing functions $f_v : \{0,1\}^k \to \{-1,1\}$ for all $v \in V$. The tester chooses an edge $(u,v) \in E$ uniformly at random and then applies the Håstad test with parameter δ to the collection $\{f_u^{\mathrm{odd}}, f_v^{\mathrm{odd}} \circ \sigma_{u\to v}\}$ where $\sigma_{u\to v} : [k] \to [k]$ is the permutation constraint on the edge (u,v) and for $x \in \{0,1\}^k$ we define $(\sigma_{u\to v}(x))_j = x_{\sigma^{-1}_{u\to v}(j)}$.

   (a) Show that completeness still works (and even better): if $\mathrm{val}(G) \geq 1 - \lambda$ then the resulting 3-linear CSP has value at least $1 - \lambda - \delta$.

   (b) Show that soundness does not work: no matter what G is, the resulting 3-linear CSP has an assignment with value at least $5/8 - \delta/4$.

References

[1] S. Arora, J. R. Lee, and A. Naor. Euclidean distortion and the sparsest cut. *J. Amer. Math. Soc.*, 21(1):1–21, 2008. Preliminary version in STOC'05.

[2] P. Enflo. On the nonexistence of uniform homeomorphisms between L_p-spaces. *Ark. Mat.*, 8:103–105 (1969), 1969.

[3] J. Matoušek. Open problems on embeddings of finite metric spaces. `http://kam.mff.cuni.cz/ matousek/metrop.ps.gz`.