Homework 1
Due 2011/2/25

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Instructions

Writeup: You must do the writeup alone. For each question, cite all references used (or write 'none') and collaborators (or write 'alone'). Explain why you needed to consult any of the references.

Collaboration: Collaboration is allowed, but limit yourselves to groups of size at most two.

References: Try not to run to reference material to answer questions (this also includes the web!). Try to think about the problem to see if you can solve it without consulting any external sources. If this fails, you may ask me for a hint, or look up any reference material.

Deadline: The deadline is strict.

Problems

1. **(no need to hand in)** Refresh your memory on some basic notions in probability such as the union bound, Markov's inequality, the Chernoff-Hoeffding bound. An especially useful formulation of the Chernoff-Hoeffding bound is the following: Let P be a probability distribution supported on some bounded range (say, $[-1,1]$). Then the average of $O(\log(1/\delta)/\varepsilon^2)$ independent samples from P is within $\pm\varepsilon$ of the true expectation of P with probability at least $1-\delta$.

2. **Fourier:** Let $f : \{0,1\}^n \to \mathbb{R}$ be a function on the Boolean cube with Fourier coefficients $\hat{f}(S)$. For each of the following functions g, compute the Fourier coefficients $\hat{g}$ in terms of $\hat{f}$. Hint: Analyze what happens for $f = \chi_S$.
   (a) $g(x) = f(x+y)$ for some $y \in \{0,1\}^n$.
   (b) $g(x) = f^{\text{odd}}(x) = \frac{1}{2}(f(x) - f(x + (1,1,\ldots,1)))$.
   (c) $g(x) = (-1)^{\langle x,y\rangle} f(x)$ for some $y \in \{0,1\}^n$ where $\langle x,y\rangle$ is defined as $\sum_i x_i y_i$.
   (d) $g(x_1,\ldots,x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for some permutation $\pi : [n] \to [n]$.
   (e) $g(x) = f(x_1,\ldots,x_k,0,\ldots,0)$ for some $0 \le k \le n$.
   (f) $g(x_1,\ldots,x_k) = f(x_1,\ldots,x_k,0,\ldots,0)$ for some $0 \le k \le n$.
   (g) Fix some sets $S \subseteq I \subseteq [n]$ and let $\bar{I} = [n] \setminus I$. Define $g : \{0,1\}^{\bar{I}} \to \mathbb{R}$ by $g(x) = \mathrm{Exp}_{y \in \{0,1\}^I}[f(x,y)\chi_S(y)]$.
   (h) $g(x_1,\ldots,x_k) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for some function $\pi : [n] \to [k]$.

3. **Variance:** Let the *variance* of $f : \{0,1\}^n \to \mathbb{R}$ be defined by $\mathrm{Var}[f] = \mathrm{Exp}[f^2] - \mathrm{Exp}[f]^2$ where the expectations are taken over a uniform choice of $x \in \{0,1\}^n$. Find an expression for the variance of f in terms of its Fourier coefficients.

4. **Estimating Fourier:** Let $f : \{0,1\}^n \to [-1,1]$ be some function. Assume we are given random samples $(x, f(x))$ for uniformly chosen $x \in \{0,1\}^n$.
   (a) Show how, given some $S \subseteq [n]$, to estimate $\hat{f}(S)$ to within $\pm\varepsilon$ with confidence $1-\delta$. How many samples are needed?
   (b) Same for $\sum_{S \subseteq [n]} \hat{f}(S)^2$.

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5. **The NAE test:** Recall that in the NAE test we pick $x, y, z \in \{0,1\}^n$ by independently choosing for each i the triple (x_i, y_i, z_i) uniformly from $\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$; we then check that $f(x)$, $f(y)$, and $f(z)$ are not all equal. Prove that the probability that this test accepts is given by
$$\frac{3}{4} - \frac{3}{4}\sum_{S\subseteq[n]}\left(-\frac{1}{3}\right)^{|S|}\hat{f}(S)^2.$$
Your proof should make use of the noise operator $T_{-\frac{1}{3}}$.

6. **Functions with no weight above level 1:**

   (a) Assume $f : \{0,1\}^n \to \{-1,1\}$ satisfies that
   $$\sum_{|S|>1}\hat{f}(S)^2 = 0,$$
   i.e., all the Fourier coefficients beyond level 1 are zero. Show that f must be a 1-junta, i.e., it depends on at most one coordinate (or in other words, f is either a constant function, a dictator function, or an anti-dictator function).

   (b) Assume now that all the Fourier coefficients beyond level 2 are zero. Is it true that f is a 2-junta?

7. **Local decodability:**

   (a) We are given oracle access to a function $f : \{0,1\}^n \to \{-1,1\}$ that is promised to be ε-close to a linear function χ_S for some unknown $S \subseteq [n]$. Show how to compute $\chi_S(x)$ with success probability at least $1-2\varepsilon$ for any given $x \in \{0,1\}^n$ using only two queries to f. Such a procedure is sometimes known as a *local decoder*.

   (b) Using the above, show how to recover S with probability 90% using as few queries as you can (assuming ε is a small enough constant).

   (c) Recall that in the Håstad test, the constant function is accepted with probability 1. Show how to use local decodability in order to transform the Håstad test into a true 3-query test for dictatorship.

8. **Poincaré inequality:** Let $f : \{0,1\}^n \to \{-1,1\}$. Define the *total influence* of f to be
$$\mathbb{I}(f) = \operatorname*{Exp}_x[\#\{i \in [n] \,:\, f(x) \neq f(x \oplus e_i)\}]$$
Show that
$$\mathbb{I}(f) \geq 4\Pr[f(x)=1]\Pr[f(x)=-1].$$
Interpret this result as a statement about the Hamming graph $G = (V = \{0,1\}^n, E)$ with edges connecting any two vertices that differ in exactly one coordinate.

9. **Convolutions:** Let $f : \{0,1\}^n \to \mathbb{R}$.

   (a) Let $f_i(x) = \frac{1}{2}(f(x) - f(x \oplus e_i))$. Show an explicit function g such that $f_i = f * g$.

   (b) Let $T_\rho f = \sum \rho^{|S|}\hat{f}(S)\chi_S$. Show an explicit function g such that $T_\rho f = f * g$.