Polarimetry of homogeneous half-spaces

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Overview

- **Scattering by homogeneous half-spaces:**
  - extension of a textbook case: Fresnel reflection coefficients;
  - inverse problem: material properties from scattered waves;
  - quantifier elimination for polarimetry with uniaxial dielectric tensor.

- **Inhomogeneous half-spaces:** vertical stratification, Leontovich boundary conditions, rough surface scattering.

![Diagram of scattering](image)

**Homogeneous scatterer**

**Inhomogeneous scatterer**
Motivation: radar imaging

Synthetic aperture radar (SAR) is an established signal processing technology that retrieves the “averaged reflectivity”.

SAR scattering model assumes independent point scatterers:
- acceptable for detection and tracking;
- not helpful for reconstruction of scatterer properties (e.g., aboveground biomass, significant wave height).
Electric $E$ and magnetic $H$ fields obey the Maxwell’s equations in material:

\[
\frac{1}{c} \partial_t H + \text{curl} \ E = 0, \quad \text{div} \ H = 0,
\]
\[
\frac{1}{c} \partial_t D - \text{curl} \ H = -\frac{4\pi}{c} j, \quad \text{div} \ D = 0.
\]

The material is characterized by dielectric $\varepsilon$ and conductivity $\sigma$ tensors:

\[
D = \varepsilon \cdot E, \quad j = \sigma \cdot E.
\]

In isotropic dielectric with losses:

\[
\varepsilon = \varepsilon I, \quad \sigma = \sigma I \implies D = \varepsilon E, \quad j = \sigma E.
\]

In vacuum:

\[
\varepsilon = 1, \quad \sigma = 0 \implies D = E, \quad j = 0.
\]
Transverse waves

- **Homogeneous isotropic perfect dielectric** is described by $\varepsilon = \varepsilon I$, where $\varepsilon$ is piecewise constant and $\sigma = 0$; hence

\[
\frac{1}{c} \frac{\partial H}{\partial t} + \text{curl } E = 0, \quad \frac{1}{c} \frac{\partial (\varepsilon E)}{\partial t} - \text{curl } H = 0, \quad \text{div}(\varepsilon E) = 0
\]

yield, away from discontinuities of $\varepsilon(\mathbf{r})$:

\[
\Delta E = \frac{\varepsilon}{c^2} \frac{\partial^2}{\partial t^2} E
\]

- **Plane waves**: solutions to the Maxwell’s equations of the form

\[
E(\mathbf{r}, t) \sim e^{i(k \cdot \mathbf{r}) - i\omega t} \quad \text{where} \quad \sqrt{\varepsilon \omega} = |k| c, \quad H = \frac{c}{\omega} k \times E
\]

(for linear problems, take either real or imaginary part)

- **Transversality**: $\text{div } E = \text{div } H = 0 \implies (k, E) = (k, H) = 0$. 

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Linear polarizations

- **Plane waves in vacuum**: solutions to the Maxwell’s equations of the form (for linear problems, take either real or imaginary part):

\[
E(r, t) \sim e^{i(k\cdot r - \omega t)}, \quad \omega = kc, \quad k = |k|, \quad H = k \times E/k
\]

with \( E \perp k, \ H \perp k, \ E \perp H, \ |E| = |H| \).

- **Planar interface**: is characterized by the normal \( n \).

- **If we have a planar interface** and \( k \parallel n \), we can define the incidence plane and two linear polarizations:
  - **Horizontal** polarization (or \( E \)-polarization)
    \[
    E(r, t) = E^A \cdot e^{i(k\cdot r - \omega t)}, \quad E^A = E_0 n \times (k/k), \quad H(r, t) = k \times E(r, t)/k.
    \]
    vector amplitude \( \perp \) incidence plane
  
  - **Vertical** polarization (or \( H \)-polarization)
    \[
    H(r, t) = H^A \cdot e^{i(k\cdot r - \omega t)}, \quad H^A = H_0 n \times (k/k), \quad E(r, t) = -k \times H(r, t)/k.
    \]
    vector amplitude \( \perp \) incidence plane
Preliminaries

Refraction-reflection problem

Polarimetry

- Any plane wave can be represented as a linear combination of two waves in basic linear polarizations (with a possible phase shift).
- We can do so for the incident and reflected waves.
- The reflecting properties can be fully described by a $2 \times 2$ matrix relating amplitudes of scattered and incident waves:

$$S = \begin{bmatrix} S_{HH} & S_{HV} \\ S_{VH} & S_{VV} \end{bmatrix}, \text{ such that } E_r^A = S \cdot E_i^A.$$  

- If we control $E_i^A$ and measure $E_r^A$ then we can determine $S$.
- Entries of $S$ may be complex $\implies$ max. 8 measurements (only 7 if the absolute phase is unavailable).
- This is how many scatterer parameters we potentially can reconstruct.
Perfect isotropic dielectric: $D = \varepsilon E$, $j = 0$

- The simplest setup to test the methodology: the field in each half-space is described by a **scalar** equation

$$
\left(\varepsilon(z) \frac{\partial^2}{\partial t^2} - \Delta\right) U = 0, \quad \varepsilon(z) = \begin{cases} 
1, & z > 0, \\
\varepsilon, & z < 0,
\end{cases}
$$

(*)

where $U$ is any field component.

- Born approximation (i.e., perturbations, or **weak scatterer**): $\delta \ll 1$,

$$
U = U^{(0)} + U^{(1)}, \quad \varepsilon = 1 + (\varepsilon - 1), \quad |U^{(1)}| \sim \delta |U^{(0)}|, \quad |\varepsilon - 1| \sim \delta.
$$

- The incident field is $U^{(0)}$; it satisfies (*) everywhere with $\varepsilon = 1$, so

$$
\left(\varepsilon(z) \frac{\partial^2}{\partial t^2} - \Delta\right) U^{(1)} = - (\varepsilon(z) - 1) \frac{\partial^2}{\partial t^2} U^{(0)}.
$$

Linearization: throw away $(\varepsilon - 1) U^{(1)} \sim \delta^2$. 
Solution in two domains

- Linearized equation: \[ \left( \frac{\partial^2}{\partial t^2} - \Delta \right) U^{(1)} = - (\varepsilon(z) - 1) \frac{\partial^2}{\partial t^2} U^{(0)}. \]
- For \( r = (x, y, z) \) we take \( U^{(0)} = u^{(0)} A e^{i(k_i r - \omega t)} \), where \( k_i = (K_i, 0, -q_i) \).
- The interface is \( (z = 0) \). Fourier representation:
  \[ U(x, y, z, t) = \hat{u}(z) e^{iK_i x - i\omega t}, \]
  where \( U \) is any field component; so the incident field is \( \hat{u}^{(0)}(z) = u^{(0)} A e^{-iq_i z} \); scattered field:
  \[
  \left( \frac{d^2}{dz^2} + q_i^2 \right) \hat{u}^{(1)}(z) = -\theta(-z) (\varepsilon - 1) k^2 \hat{u}^{(0)}(z)
  = -\theta(-z) (\varepsilon - 1) k^2 u^{(0)} A e^{-iq_i z},
  \]
  where \( \theta(\cdot) \) is a step function. For \( z > 0 \), \( \hat{u}^{(1)}(z) \) is also a plane wave; for \( z < 0 \), \( \hat{u}^{(1)}(z) \) is a forced oscillation; note the resonance.
- “General solution without RHS + particular solution with RHS”:
  \[
  \hat{u}^{(1)} = \begin{cases} 
  u^{(0)} A Be^{iq_i z}, & z > 0 \\
  u^{(0)} A (Az + C) e^{-iq_i z}, & z < 0,
  \end{cases} \quad A = -\frac{-ik^2(\varepsilon-1)}{q_i}.
  \]
Origin of the linearly growing term

- The linearly growing term $Aze^{-iq_iz}$ is unphysical.
- This term is an artifact of the Born approximation, in particular — ignoring the change of the vertical wavenumber:

$$k = \sqrt{\varepsilon} \frac{\omega}{c}, \quad q_t^{(\text{exact})} = \sqrt{\varepsilon \frac{\omega^2}{c^2} - K_i^2}, \quad q_t^{(\text{Born})} = q_i = \sqrt{\frac{\omega^2}{c^2} - K_i^2}.$$

- For $z \to 0$, the difference between the two periodic solutions is

$$e^{-iq_t^{(\text{exact})}z} - e^{-iq_t^{(\text{Born})}z} = e^{-iq_t^{(\text{Born})}z} \left( e^{-i(q_t^{(\text{exact})} - q_t^{(\text{Born})})z} - 1 \right)$$

$$\approx e^{-iq_t^{(\text{Born})}z} ( - i (q_t^{(\text{exact})} - q_t^{(\text{Born})})z )$$

$$\approx e^{-iq_t^{(\text{Born})}z} . (-i) \frac{1}{2} \frac{(\omega/c)^2}{q_i} (\varepsilon - 1) z.$$

- Hence, the transmitted part of the Born solution is valid in the vicinity of the interface (important for interface conditions).
Interface conditions

- Needed where \( \text{div} \ D = \text{div}(\varepsilon \cdot E) \) does not exist.
- Relate the free-space (F) and material (M) sides of the interface

\[
E_{x,y} |_{(F)} = E_{x,y} |_{(M)}, \quad H_{x,y} |_{(F)} = H_{x,y} |_{(M)}.
\]

- For plane waves \( U = \hat{U}(z) e^{iK_i x - i\omega t} \), the interface conditions reduce to two conditions for a single variable, typically — the component that is normal to the incidence plane (xz):
  - Horizontal polarization, \( E = (0, E_y, 0), H = (H_x, 0, H_z) \):
    \[
    \hat{E}_y |_{(F)} = \hat{E}_y |_{(M)}, \quad \frac{d\hat{E}_y}{dz} |_{(F)} = \frac{d\hat{E}_y}{dz} |_{(M)}.
    \]
  - Vertical polarization, \( H = (0, H_y, 0), E = (E_x, 0, E_z) \):
    \[
    \frac{d\hat{H}_y}{dz} |_{(F)} = \left( \varepsilon^{-1} \frac{d\hat{H}_y}{dz} \right) |_{(M)}, \quad \hat{H}_y |_{(F)} = \hat{H}_y |_{(M)}.
    \]
Reflection coefficients

- We linearize the Interface conditions and use universal notation $\hat{u}$ for $\hat{E}_y$ and $\hat{H}_y$ according to the polarization:
  
  **both polarizations:**
  
  \[
  \left. \hat{u}^{(1)} \right|_{(F)} = \left. \hat{u}^{(1)} \right|_{(M)},
  \]

  **horizontal:**
  
  \[
  \left. \frac{d\hat{u}^{(1)}}{dz} \right|_{(F)} = \left. \frac{d\hat{u}^{(1)}}{dz} \right|_{(M)},
  \]

  **vertical, linearized:**
  
  \[
  \left. \frac{d\hat{u}^{(1)}}{dz} \right|_{(F)} = \left. \frac{d\hat{u}^{(1)}}{dz} \right|_{(M)} - (\varepsilon - 1) \left. \frac{d\hat{u}^{(0)}}{dz} \right|_{z=0}
  \]

- From $\hat{u}^{(1)} = \begin{cases} \left( u^{(0)A} Be^{iq_i z} \right. & z > 0, \\ \left. u^{(0)A} \left( -\frac{-ik^2(\varepsilon - 1)}{q_i} z + C \right) e^{-iq_i z} \right. & z < 0, \end{cases}$ we obtain:
  
  **horizontal:** $B \equiv S_{HH} = - (\varepsilon - 1) \frac{k^2}{4q_i^2},$

  **vertical:** $B \equiv S_{VV} = (\varepsilon - 1) \left( \frac{1}{2} - \frac{k^2}{4q_i^2} \right) = S_{HH} + \frac{\varepsilon - 1}{2}.$

- Both expressions coincide with the Fresnel formulas as $|\varepsilon - 1| \to 0.$
Summary for isotropic dielectric

- Test case of the methodology.
- Direct problem:
  - Real-valued reflection coefficients.
  - Obtained correct asymptotics for reflection coefficients.
  - Single scatterer parameter: \((\varepsilon - 1)\).
  - No cross-polarized scattering: can satisfy the differential equations and interface conditions within a single linear polarization.
- Inverse problem:
  - Inversion is straightforward: \((\varepsilon - 1) = -\frac{4q_i^2}{k^2} S_{HH}\).
  - 2 out of 4 entries of the matrix \(S\).
  - The data has only 1 out of 8 degrees of freedom because
    \[
    \frac{S_{VV}}{S_{HH}} = Q = \frac{K_i^2 - q_i^2}{k^2}
    \]
    does not depend on the properties of the scatterer.
Isotropic lossy dielectric

- Finite isotropic conductivity: \( j = \sigma E \).
- For \( E = \tilde{E} e^{-i\omega t} \), and similarly for \( H \) and \( D \):

\[
\text{curl } \tilde{H} = -ik\tilde{D} + \frac{4\pi}{c} \sigma \tilde{E} = -ik \left( \varepsilon + i \frac{4\pi}{\omega} \sigma \right) \tilde{E} = -ik\varepsilon' \tilde{E}.
\]

- Use \( \varepsilon' \) instead of \( \varepsilon \) for the case of perfect dielectric.
- Need \( \sigma/\omega \ll 1 \) to use in Born approximation.
- Direct problem:
  - two parameters of the scatterer: \((\varepsilon - 1)\) and \(\sigma/\omega\);
  - reflection coefficients are complex;
  - otherwise similar to the perfect dielectric.

- Inversion: \((\varepsilon' - 1) = -\frac{4q_i^2}{k^2} S_{HH}\) (similar to the case of perfect dielectric). Note that \(S_{VV}/S_{HH}\) is independent of the material properties \(\implies\) need the absolute phase to detect \(\sigma\).
Dielectric tensor

- Anisotropic material: \( D_i = \varepsilon_{ij}E_j, \varepsilon \neq \varepsilon I \).
- From energy considerations, \( \varepsilon \) is symmetric.
- For a real symmetric matrix \( \varepsilon \), choose \((x', y', z')\) such that
  \[
  D_i = \text{diag}(\varepsilon_{x'}, \varepsilon_{y'}, \varepsilon_{z'}) \varepsilon_{ij}E_j.
  \]

Uniaxial model: two (rather than three) independent parameters (i.e., matrix eigenvalues): \( \varepsilon_{x'} = \varepsilon_{y'} = \varepsilon_\perp; \varepsilon_{z'} = \varepsilon_\parallel; e_z' \) is the optical axis. We still require \( |\varepsilon_\perp - 1| \ll 1 \) and \( |\varepsilon_\parallel - 1| \ll 1 \).

Let \( e_z' = (\alpha, \beta, \gamma) \), \( \alpha^2 + \beta^2 + \gamma^2 = 1 \), and \( \zeta = \varepsilon_\parallel - \varepsilon_\perp \), then
\[
\varepsilon_{xx} = \varepsilon_\perp + \alpha^2 \zeta, \quad \varepsilon_{yy} = \varepsilon_\perp + \beta^2 \zeta, \quad \varepsilon_{zz} = \varepsilon_\perp + \gamma^2 \zeta,
\]
\[
\varepsilon_{xy} = \varepsilon_{yx} = \alpha \beta \zeta, \quad \varepsilon_{xz} = \varepsilon_{zx} = \alpha \gamma \zeta, \quad \varepsilon_{yz} = \varepsilon_{zy} = \beta \gamma \zeta.
\]

For \( \eta = \varepsilon^{-1} \), we have, up to the first order:
\[
\eta_{xx} = 1/\varepsilon_{xx}, \quad \eta_{yy} = 1/\varepsilon_{yy}, \quad \eta_{zz} = 1/\varepsilon_{zz},
\]
\[
\eta_{xy} = \eta_{yx} = -\varepsilon_{xy}, \quad \eta_{xz} = \eta_{zx} = -\varepsilon_{xz}, \quad \eta_{yz} = \eta_{zy} = -\varepsilon_{yz}.
\]
Governing equations

- We can reduce the Maxwell’s equations

\[
- \frac{d\hat{E}_y}{dz} = ik\hat{H}_x, \quad - \frac{d\hat{H}_y}{dz} = -ik\hat{D}_x,
\]

\[
\frac{d\hat{E}_x}{dz} - iK_i\hat{E}_z = ik\hat{H}_y, \quad \frac{d\hat{H}_x}{dz} - iK_i\hat{H}_z = -ik\hat{D}_y,
\]

\[
iK_i\hat{E}_y = ik\hat{H}_z, \quad iK_i\hat{H}_y = -ik\hat{D}_z,
\]

to the following system for \(\hat{E}_y\) and \(\hat{H}_y\):

\[
\left( \eta_{yy} \frac{d^2}{dz^2} + k^2 - K_i^2 \eta_{yy} \right) \hat{E}_y = - k \left( i \eta_{xy} \frac{d}{dz} + K_i \eta_{yz} \right) \hat{H}_y,
\]

\[
\left( \eta_{xx} \frac{d^2}{dz^2} + k^2 - K_i^2 \eta_{zz} \right) \hat{H}_y - 2i \eta_{xz} K_i \frac{d\hat{H}_y}{dz} = - \frac{i}{k} \left( i \eta_{yz} K_i - \eta_{xy} \frac{d}{dz} \right) \left( \frac{d^2 \hat{E}_y}{dz^2} - K_i^2 \hat{E}_y \right),
\]

and substitute \(\hat{E}_x\) and \(\hat{H}_x\) expressed via \(\hat{E}_y\) and \(\hat{H}_y\) into

\[
\hat{E}_{x,y} \bigg|_{(F)} = \hat{E}_{x,y} \bigg|_{(M)}, \quad \hat{H}_{x,y} \bigg|_{(F)} = \hat{H}_{x,y} \bigg|_{(M)}.
\]
In Born approximation, the system of two equations decouples in a different way for different polarizations of the incident field.

For example, when $\hat{E}_i = (0, E^{(0)A} e^{-iq_iz}, 0)$, then $|\hat{H}_y| \sim |\hat{H}^{(1)}| \ll |E^{(0)A}|$, and in

$$
\left( \eta_{yy} \frac{d^2}{dz^2} + k^2 - K_i^2 \eta_{yy} \right) \hat{E}_y = -k \left( i \eta_{xy} \frac{d}{dz} + K_i \eta_{yz} \right) \hat{H}_y,
$$

we drop RHS $\sim \delta^2$ and then linearize LHS to obtain

$$
\left( \frac{d^2}{dz^2} + q_i^2 \right) \hat{E}_y^{(1)}(z) = -\theta(-z)(\varepsilon - 1)k^2 E^{(0)A} e^{-iq_iz} + |\text{ICs}} \rightarrow S_{HH},
$$

whereas for $\hat{H}_i = (0, H^{(0)A} e^{-iq_iz}, 0)$ and $|\hat{E}_y| \sim |\hat{E}^{(1)}| \ll |H^{(0)A}|$ we linearize both sides:

$$
\left( \frac{d^2}{dz^2} + q_i^2 \right) \hat{E}_y^{(1)} = -k \left( i \eta_{xy} \frac{d}{dz} + K_i \eta_{yz} \right) \hat{H}_y^{(0)} + |\text{ICs}} \rightarrow S_{HV}.
$$
Direct and inverse problems

- Direct \((\alpha, \beta, \gamma, \xi, \zeta) \rightarrow S\) and inverse \(S \rightarrow (\alpha, \beta, \gamma, \xi, \zeta)\) problems:

\[
S_{\text{HH}} = -\frac{1}{4} \frac{k^2}{q_i^2} (\xi + \beta^2 \zeta), \quad S_{\text{HV}} = \frac{1}{4} \frac{k^2}{q_i} \left(\alpha + \frac{K_i}{q_i} \gamma\right) \zeta \beta,
\]

\[
S_{\text{VV}} = \frac{1}{4} \left( (\xi + \alpha^2 \zeta) - \frac{K_i^2}{q_i^2} (\xi + \gamma^2 \zeta) \right), \quad S_{\text{VH}} = -\frac{1}{4} \frac{k^2}{q_i} \left(\alpha - \frac{K_i}{q_i} \gamma\right) \zeta \beta,
\]

\[
1 = \alpha^2 + \beta^2 + \gamma^2 \quad \text{(components of } e_{z'})\nonumber
\]

where \(\frac{k}{q_i}\) and \(\frac{K_i}{q_i}\) — parameters, \(k^2 = K_i^2 + q_i^2\); \(\xi = \varepsilon_\perp - 1\), \(\zeta = \varepsilon_\parallel - \varepsilon_\perp\).

- Inverse problem: 4 polynomials of 3rd degree and 1 polynomial of 2nd degree.

- Degrees of freedom vs. orientation of optical axis \((S_{ij} \in \mathbb{R})\):
  - 2 if \(\beta = 0\) because \(S_{\text{HV}} = S_{\text{VH}} = 0\);
  - 3 if \((\alpha = 0 \text{ and } \beta \gamma \neq 0)\) OR \((\gamma = 0 \text{ and } \alpha \beta \neq 0)\) because \(S_{\text{HV}} = S_{\text{VH}} = 0\);
  - 4 if \(\alpha \beta \gamma \neq 0\).
Solvability of the inverse problem

- Proposition: The system of equations can be solved with respect to $\varepsilon_\perp$, $\varepsilon_\parallel$, $\alpha$, $\beta$, and $\gamma$ for the given $S_{HH}$, $S_{VV}$, $S_{HV}$, $S_{VH}$, and $\theta_{\text{inc}}$ if and only if
  \[(S_{VV} + VS_{HH})^2 \geq 4WS_{HV}S_{VH}, \quad (**)
  \]
  where
  \[W = \frac{q_i^2 - K_i^2}{k^2} = \cos^2 \theta_{\text{inc}} - \sin^2 \theta_{\text{inc}} = \cos 2\theta_{\text{inc}}.\]

- If $\alpha = 0$ then $S_{HV} = S_{VH}$, and for $\theta_{\text{inc}} > \pi/4$ we have $W < 0 \implies$ RHS $< 0$ in (**)$\implies$ the problem always has a solution.

- So, (**') puts an additional constraint on the values of the reflection coefficients and thus implies a limitation of solvability of the linearized inverse problem.

- Derivation of (**') has taken several weeks ...
Dear Mikhail,

I applied a general method and obtained the following result (in about 3 seconds):

\begin{align*}
0 &= -s_1 - \frac{1}{4}k^2(x+b^2z), \\
0 &= -s_2 + \frac{1}{4}((x+a^2z)-K^2(x+g^2z)), \\
0 &= -s_3 - \frac{1}{4}k(a-Kg)z^2b, \\
0 &= -s_4 + \frac{1}{4}k(a+Kg)z^2b, \\
0 &= -1 + a^2 + b^2 + g^2, \\
0 &= -k^2 + K^2 + 1
\end{align*}

iff

\begin{align*}
K^4s_1^2 - 2K^4s_1s_2 + K^4s_2^2 + 4K^4s_3s_4 - 2K^2s_1^2 \\
+ 2K^2s_2^2 + s_1^2 + 2s_1s_2 + s_2^2 - 4s_3s_4 &> 0
\end{align*}
Summary for anisotropic dielectric

- **Direct problem:**
  - Real-valued reflection coefficients.
  - Multiple scatterer parameters.
  - Can produce cross-polarized scattering.

- **Inverse problem:**
  - All 4 entries of the matrix $S$.
  - The data has 2 to 4 (out of 8) degrees of freedom.
  - Special condition of solvability (because the material parameters must be real).
  - If we allow uniaxial conductivity, we can get all 8 degrees of freedom. Separate solvability problems for $\text{Re}(S')$ and for $\text{Im}(S')$.

- Biaxial dielectric — may serve as an example of underdetermined system of equations (not done yet).
Main deficiency of homogeneous half-space models: only specular reflection, no backscattering.

Backscattering is the primary configuration for remote sensing.

Options to obtain backscattered signal:
- horizontally inhomogeneous material;
- horizontally inhomogeneous boundary conditions;
- horizontally inhomogeneous shape of the interface.
For $\varepsilon = \varepsilon(x, y)$, Maxwell’s equations reduce to

$$
\Delta E - \text{grad div } E = \frac{\varepsilon}{c^2} \frac{\partial^2}{\partial t^2} E, \quad \text{div}(\varepsilon E) = 0.
$$

Plane wave $E \sim e^{i(k_r r - \omega t)}$ is not a solution (try $\varepsilon = 1 + a \sin(2K_ix)$).

Linearization is needed to move forward.

We consider an acoustic problem $\Delta U = \frac{\varepsilon}{c^2} \frac{\partial^2}{\partial t^2} U$ for $U(r, t)$ where $U = U^{(0)} + U^{(1)}$ and $\varepsilon = \varepsilon^{(0)} + \varepsilon^{(1)}(x, y)$, $|U^{(1)}| \ll |U^{(0)}|$, $|\varepsilon^{(1)}| \ll \varepsilon^{(0)}$, whereas $|\varepsilon^{(0)} - 1|$ is not necessarily small.

Linearization yields $U^{(0)}$ as the Fresnel solution for the horizontal polarization:

$$
U^{(0)} = \begin{cases} 
U_i^{(0)} + U_r^{(0)}, & z > 0, \\
U_t^{(0)}, & z < 0.
\end{cases}
$$
Linearization

- In Fourier domain:

\[
\varepsilon^{(1)}(x, y) = \frac{1}{(2\pi)^2} \int \int \hat{\varepsilon}^{(1)}(K_x, K_y)e^{i(K_x x + K_y y)} dK_x dK_y,
\]

we obtain

\[
\left( \frac{d^2}{dz^2} + q^2 \right) \hat{u}^{(1)} = 0, \quad z > 0,
\]

\[
\left( \frac{d^2}{dz^2} + q'^2 \right) \hat{u}^{(1)} = -k^2 u^{(0)A} \hat{\varepsilon}^{(1)}_B e^{-iq'_r z}, \quad z < 0,
\]

where \( \hat{u}^{(1)} = \hat{u}^{(1)}(K_x, K_y, z), \quad \hat{\varepsilon}^{(1)}_B = \hat{\varepsilon}^{(1)}(K_x - K_i, K_y), \quad q^2 = k^2 - K_x^2 - K_y^2, \quad q'^2 = k^2 \varepsilon^{(0)} - K_x^2 - K_y^2, \quad q'_r = k^2 \varepsilon^{(0)} - K_i^2. \)

- Backscattering for \( z > 0 \) means \( K_x = -K_i, K_y = 0 \), i.e.,

\[
\hat{\varepsilon}^{(1)}_B \equiv \hat{\varepsilon}^{(1)}(-2K_i, 0). \] This is the Bragg harmonic of \( \varepsilon(x, y) \).
Reflection coefficients

General solution:

\[ \hat{u}^{(1)} = u^{(0)A} \cdot \begin{cases} 
Be^{iqz}, & z > 0, \\
Ce^{-iq'z} + A_1 e^{-iq'_r z}, & z < 0 \text{ and } q' \neq q'_r, \\
Ce^{-iq'z} + A_2 z e^{-iq'_r z}, & z < 0 \text{ and } q' = q'_r.
\]

Interface conditions: continuity for \( \hat{u}^{(1)} \) and \( d\hat{u}^{(1)}/dz \) at \( z = 0 \).

Reflection coefficients:

\[ B = \begin{cases} 
-2\hat{\epsilon}_{B}^{(1)} \frac{k^2 q_i}{(q' + q)(q'_r + q')(q'_r + q_i)}, & q' \neq q'_r, \\
-2\hat{\epsilon}_{B}^{(1)} \frac{k^2 q_i}{(q'_r + q)2q'_r(q'_r + q_i)}, & q' = q'_r.
\]

\( B \) is insensitive to the resonance in the material (i.e., two expressions for \( B \) coincide if \( q' = q'_r \)).
Summary for inhomogeneous material

- Scalar problem for inhomogeneous isotropic dielectric.
- Linearization of $\varepsilon$ about $\varepsilon^{(0)}$.
- The scattering coefficient is proportional to $\hat{\varepsilon}_B^{(1)}$.
- Vector problem (i.e., Maxwell’s equations rather than wave equation) is huge (check [GST, 2017]). It demonstrates depolarization if $K_y \neq 0$. No depolarization for backscattering.
- Not done yet: inhomogeneous anisotropic material. We expect to obtain depolarization in backscattering.
Polarization: additional information about the target

- Polarimetry is useful for segmentation/classification of terrains:

  - RAMSES (Airborne SAR)

  - Polarimetry reveals, e.g.:
    - forested land by high HV;
    - bare ground and grass by high HH-VV;

  although the models are still semi-empirical.

https://earth.esa.int/documents/10174/669756/Urban_Classification_3Drendering.pdf
Alternatives for refraction-reflection problem

- We can consider reflection-only problem because we are not interested in the field below the interface.
  - Advantage: simpler setup.
  - Disadvantage: boundary properties should represent bulk properties.

- Possible options:
  - Variable boundary conditions on a plane surface (example: Leontovich boundary condition).
  - Homogeneous boundary conditions on a non-plane surface (i.e., rough surface scattering).
  - Combination of the two inhomogeneities.
Leontovich boundary condition

Let $\varepsilon = \varepsilon^{(0)} = \text{const}$. In Fourier domain:

$$\hat{u}(K_x, K_y, z) = e^{i\omega t} \iiint U(x, y, z, t) e^{-i(K_x x + K_y y)} \, dx \, dy,$$

consider the regular reflection-refraction problem

$$\hat{u}(K_x, K_y, z) = \begin{cases} 
\hat{u}_i^A(K_x, K_y) e^{-iqz} + \hat{u}_r^A(K_x, K_y) e^{iqz}, & z > 0, \\
\hat{u}_t^A(K_x, K_y) e^{-iq'z}, & z < 0,
\end{cases}$$

where $q^2 = k^2 - K_x^2 - K_y^2$, $q'^2 = k^2 \varepsilon^{(0)} - K_x^2 - K_y^2$. If $\hat{u}$ and $\partial\hat{u}/\partial z$ are continuous at $z = 0$, then we can show that

$$\left. \frac{\partial \hat{u}(K_x, K_y, z)}{\partial z} \right|_{z=+0} = -i\sqrt{\varepsilon^{(0)} k^2 - (K_x^2 + K_y^2)} \cdot \hat{u}(K_x, K_y, z) \bigg|_{z=+0}.$$

Hence, in the coordinate domain, the relation between $U$ and $\partial U/\partial z$ will be non-local.
Inhomogeneous half-spaces
Reflection-only setups

Leontovich boundary condition, cont’d

- For $\varepsilon^{(0)} \gg 1$, we can simplify $\sqrt{\varepsilon^{(0)} k^2 - (K_x^2 + K_y^2)} \approx \sqrt{\varepsilon^{(0)} k}$, so

\[
\frac{\partial \hat{u}(K_x, K_y, z)}{\partial z} \bigg|_{z=+0} = -i\sqrt{\varepsilon^{(0)} k} \cdot \hat{u}(K_x, K_y, z) \bigg|_{z=+0}
\]

$\Downarrow$

\[
\frac{\partial U(x, y, z, t)}{\partial z} \bigg|_{z=+0} = -i\sqrt{\varepsilon^{(0)} k} \cdot U(x, y, z, t) \bigg|_{z=+0}.
\]

These are local boundary conditions, called Leontovich conditions.

- Generalization:

\[
\frac{\partial U(x, y, z, t)}{\partial z} \bigg|_{z=+0} = -i\sqrt{\varepsilon(x, y) k} \cdot U(x, y, z, t) \bigg|_{z=+0}, \quad \varepsilon(x, y) \gg 1.
\]
We don’t need a solution for $z < 0$ because we have boundary conditions at $z = 0$.

With $\varepsilon(x, y) = \varepsilon^{(0)} + \varepsilon^{(1)}(x, y)$, $|\varepsilon^{(1)}| \ll |\varepsilon^{(0)}|$, and $U = U^{(0)} + U^{(1)}$, $U^{(0)} = \hat{u}_i^A e^{i(K_i x - q_i z) - i\omega t}$, $|U^{(1)}| \ll |U^{(0)}|$, we can obtain

$$
\frac{\hat{u}_r^A(K_x, K_y)}{\hat{u}_i^A} = -\frac{\hat{\varepsilon}_B^{(1)}}{(\varepsilon^{(0)})^{3/2}} \frac{q_i}{k}.
$$

— this is the reflection coefficient $\propto \hat{\varepsilon}_B^{(1)} = \hat{\varepsilon}^{(1)}(K_x - K_i, K_y)$.

The reflection coefficient coincides with that obtained in the reflection-refraction problem for $\varepsilon^{(0)} \gg 1$.

Need more validation.

Possible next step: vector problem and anisotropic $\varepsilon^{(1)}$. 
Rough surface scattering

- Intensely studied topic, important for remote sensing of sea surface (see, e.g., [Beckmann & Spizzichino, 1963], [Bass & Fuks, 1979], [Voronovich, 1998], [Bruno et al., 2002]).

- The simplest setting: Dirichlet problem at \( z = h(x, y) \) for the scalar (Helmholtz) equation (strip \( e^{-i\omega t} \) factor):
  
  \[
  (\Delta + k^2)(u^{(0)}_i + u^{(0)}_r + u^{(1)}) = 0, \quad z > h(x, y),
  \]
  
  \[
  (u^{(0)}_i + u^{(0)}_r + u^{(1)}) = 0, \quad z = h(x, y).
  \]

- In Fourier domain:
  
  \[
  \hat{u}^{(1)}(K_x, K_y, z) = \int \int u^{(1)}(x, y, z)e^{-i(K_xx + K_yy)}\,dx\,dy,
  \]
  
  \[
  \hat{h}(K_x, K_y) = \int \int h(x, y)e^{-i(K_xx + K_yy)}\,dx\,dy,
  \]

  we have for the first order field:

  \[
  \left( \frac{d^2}{dz^2} + q^2 \right)\hat{u}^{(1)} = 0, \quad \text{where} \quad q^2 = k^2 - K_x^2 - K_y^2.
  \]
Rough surface scattering, cont’d

- Linearization of the domain and boundary conditions for $|q_i h| \ll 1$:

$$u^{(1)}(x, y, 0) = - \left( u_i^{(0)} + u_r^{(0)} \right)_{z=h(x,y)},$$

$$\hat{u}_i^{(0)}(K_x, K_y) \bigg|_{z=h(x,y)} = \int\int u_i^{(0)A} e^{i(K_x - q_i z)} \bigg|_{z=h(x,y)} e^{-i(K_x x + K_y y)} dx dy$$

$$\approx \int\int u_i^{(0)A} e^{iK_x x} (1 - iq_i h(x, y)) e^{-i(K_x x + K_y y)} dx dy$$

$$= u_i^{(0)A} \left( (2\pi)^2 \delta(K_x - K_i) \delta(K_y) - iq_i \hat{h}_B \right),$$

where $\hat{h}_B = \hat{h}(K_x - K_i, K_y)$, and similarly for $\hat{u}_r^{(0)}$, so (***) yields

$$\hat{u}^{(1)}(K_x, K_y, 0) = u_i^{(0)A} \cdot 2iq_i \hat{h}_B.$$

- Using $\hat{u}^{(1)} = u_i^{(0)A} B e^{iq_z}$ we find reflection coefficient to be $2iq_i \hat{h}_B$.

- Vector problem — no depolarization [Voronovich, 1998].
Summary

This talk dealt with the following topics:

- Need for physics-based scattering models for inverse problems in radar imaging.
- Polarimetry and degrees of freedom of the scattering matrix.
- Half-space models of scatterer: horizontally homogeneous and horizontally inhomogeneous.
- Single-domain models: Leontovich boundary condition and rough surface scattering.

Not mentioned:

- pulsed/modulated signal, interaction between surface harmonics, speckle, coherence, non-linearities in the scatterer, etc.

THANK YOU!
2 d.o.f. if $\beta = 0$; 3 d.o.f. if $\alpha = 0$ or $\gamma = 0$.
Limited solvability: $\theta_{\text{inc}} < \pi/4$; $\alpha = 0 \iff S_{\text{HV}} = S_{\text{VH}}$

$\theta = 2\pi/9$, $\varepsilon_\parallel$

$S_{\text{HV}} = 0.0181$

$S_{\text{HV}} = 0.0302$

$S_{\text{HV}} = 0.0423$

$S_{\text{HV}} = 0.0544$

$S_{\text{HV}} = 0.0665$

$S_{\text{HV}} = 0.0786$

$S_{\text{HV}} = 0.0907$

$S_{\text{HV}} = 0.103$
Limited solvability: blow-up
Unlimited solvability: $\theta_{\text{inc}} > \pi/4$; $\alpha = 0 \implies S_{HV} = S_{VH}$