Asymptotic valued differential fields

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For the purposes of this talk, all fields in sight have characteristic 0.
Hardy fields

- field of germs at infinity of real valued functions that is closed under differentiation

Examples:
- $\mathbb{Q}$, $\mathbb{R}(x)$, $\mathbb{R}(\sqrt{x})$, $\mathbb{R}(x, e^x, \log x)$, Hardy's LE-functions

DEFINABLE UNARY FUNCTIONS IN AN O-MINIMAL EXPANSION OF $\mathbb{R}$

Natural ordering on germs induces valuation with valuation ring $O = \{[f] : |f| \leq c, \text{eventually}\}$

Satisfy "L'Hôpital's Rule at $\infty$":

$$\lim_{x \to \infty} f(x)g(x) = \lim_{x \to \infty} f'(x)g'(x)$$
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- satisfy “L’Hôpital’s Rule at $\infty$”:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$
Transseries

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- \(\mathbb{T}\) is real closed and closed under exponentiation, integration, composition, compositional inversion, and resolution of certain ODEs
  - example series:
    \[
    7e^{e^x + e^{x/2} + e^{x/4} + \cdots} - 3e^{x^2} + 5x\sqrt{2} - (\log x)^\pi + 42 + x^{-1} + x^{-2} + \cdots + e^{-x}
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- satisfies valuation analogue of "L'Hôpital's Rule at \(\infty\)"
- introduced by Écalle in proving Dulac's conjecture and Dahn–Göring in studying models of the reals with exponentiation
- studied also by Aschenbrenner, van den Dries, and van der Hoeven:
  - axiomatization
  - model completeness in ordered valued differential field language
  - quantifier elimination in language expanded by three extra predicates
Theorem (Ax–Kochen, Ershov)

Let $K_1$ and $K_2$ be henselian valued fields. Then

$$K_1 \equiv K_2 \iff k_1 \equiv k_2 \text{ and } \Gamma_1 \equiv \Gamma_2.$$
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Tools:

1. maximal immediate extensions of $K$ are isomorphic over $K$
2. $K$ is henselian $\iff$ it is algebraically maximal
3. $K$ has a henselization
Valued fields

A *valued field* is a field \( K \) with a surjective map \( v: K \to \Gamma \cup \{\infty\} \), where \( \Gamma \) is an ordered abelian group and \( \Gamma < \infty \), satisfying:

1. \[ v(x) = \infty \iff x = 0; \]
2. \[ v(xy) = v(x) + v(y); \]
3. \[ v(x + y) \geq \min\{v(x), v(y)\}. \]
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1. $v(x) = \infty \iff x = 0$;
2. $v(xy) = v(x) + v(y)$;
3. $v(x + y) \geq\!\!\!\min\{v(x), v(y)\}$.

Notation:

- write $f \preceq g$ if $vf \geq vg$ and $f \prec g$ if $vf > vg$
- $\mathcal{O} := \{f : f \preceq 1\}$ is the valuation ring
- $\mathfrak{o} := \{f : f \prec 1\}$ is the (unique) maximal ideal of $\mathcal{O}$
- $k := \mathcal{O}/\mathfrak{o}$ is the residue field
Differential fields

A differential field is a field $K$ with a map $\partial: K \to K$ satisfying

1. $\partial(f + g) = \partial(f) + \partial(g)$;
2. $\partial(fg) = \partial(f)g + f\partial(g)$.
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1. $\partial(f + g) = \partial(f) + \partial(g)$;
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Notation:

- $f' := \partial(f)$
- $C := \{f : f' = 0\}$ is the constant field of $K$
- $K\{Y\} := K[Y, Y', Y'', \ldots]$ is the differential polynomial ring over $K$
Basic properties

- Assume $K$ has *small derivation*: $\partial \mathcal{O} \subseteq \mathcal{O}$.
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- $K$ is maximal if it has no proper immediate extensions
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- an extension of $K$ is *immediate* if it has the same value group and residue field as $K$

- $K$ is *maximal* if it has no proper immediate extensions

- $K$ is *d-algebraically maximal* if it has no proper $d$-algebraic immediate extensions
Differential-henselianity

- $K$ is d-henselian if for all $P \in \mathcal{O}\{Y\}$,  
  \[ \text{deg } \overline{P} = 1 \implies P \text{ has a zero in } \mathcal{O} \]
Differential-henselianity

- $K$ is d-henselian if for all $P \in \mathcal{O}\{Y\}$,
  
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- $k$ is linearly surjective if every $1 + a_0 Y + a_1 Y' + \cdots + a_r Y^{(r)}$, $a_i \in k$, $a_r \neq 0$, has a zero in $k$

- Note: $K$ is d-henselian $\implies k$ is linearly surjective
Uniqueness of maximal immediate extensions

Theorem (Aschenbrenner–van den Dries–van der Hoeven)

There is a valued differential field with continuum-many maximal immediate extensions that are pairwise nonisomorphic.
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Conjecture (Aschenbrenner–van den Dries–van der Hoeven)

If \( k \) is linearly surjective, then any two maximal immediate extensions of \( K \) are isomorphic over \( K \).
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**Conjecture (Aschenbrenner–van den Dries–van der Hoeven)**

If $k$ is linearly surjective, then any two maximal immediate extensions of $K$ are isomorphic over $K$.

This has been proven for monotone $K$ by Aschenbrenner, van den Dries, and van der Hoeven, and for $K$ whose value group has finite archimedean rank by van den Dries and PC.
Uniqueness of maximal immediate extensions for asymptotic fields

*K* is *asymptotic* if for all nonzero \( f, g \prec 1, \)

\[
\frac{f}{g} \prec 1 \iff \frac{f'}{g'} \prec 1.
\]

Note that then \( C \subseteq O. \)
Uniqueness of maximal immediate extensions for asymptotic fields

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Note that then $C \subseteq \mathcal{O}$.

**Theorem (PC)**

*Suppose K is asymptotic and k is linearly surjective. Then any two maximal immediate extensions are isomorphic over K.*
Differential-henselianity and differential-algebraic maximality

**Theorem (Aschenbrenner–van den Dries–van der Hoeven)**

If $k$ is linearly surjective and $K$ is $d$-algebraically maximal, then $K$ is $d$-henselian.

The converse is false in general, even in the monotone case.

**Theorem (PC)**

If $K$ is $d$-henselian and asymptotic, then it is $d$-algebraically maximal. This was first proven in the monotone case by Aschenbrenner, van den Dries, and van der Hoeven.
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Differential-henselizations

$L$ is a d-henselization of $K$ if:

1. it is a d-henselian immediate extension of $K$;
2. it embeds over $K$ into every d-henselian immediate extension of $K$. 

Theorem (Aschenbrenner–van den Dries–van der Hoeven)

If $K$ is asymptotic and $k$ is linearly surjective, then $K$ has a minimal d-henselian d-algebraic immediate extension.

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Suppose $K$ is asymptotic and $k$ is linearly surjective. Then:

1. any two maximal immediate extensions of $K$ are isomorphic over $K$;
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Proof sketch of (2)

**Theorem (PC)**

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Key properties of d-henselian asymptotic fields:

- Each $P \in \mathcal{O}[Y, Y', \ldots, Y^{(r)}]$ does not have $r + 2$ distinct zeroes in a certain configuration.
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- If $P \in \mathcal{O}\{Y\}$ with $\deg \overline{P} = 1$, and $E$ is an immediate extension of $K$, then $P$ has the same zeroes in $\mathcal{O}_E$ as in $\mathcal{O}$. 


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Proof sketch of theorem:

1. take $f$ in an immediate extension of $K$, so $f$ is the pseudolimit of a pseudocauchy sequence $(f_\rho)$ over $K$;
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Proof sketch of theorem:

1. take \( f \) in an immediate extension of \( K \), so \( f \) is the pseudolimit of a pseudocauchy sequence \( (f_\rho) \) over \( K \);
2. find minimal \( P \) such that \( P \in \mathcal{O}\{Y\} \), \( P(f_\rho) \rightrightarrows 0 \), and \( P(f) = 0 \);
Proof sketch of (2)

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If $K$ is d-henselian and asymptotic, then it is d-algebraically maximal.

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Proof sketch of theorem:

1. take $f$ in an immediate extension of $K$, so $f$ is the pseudolimit of a pseudocauchy sequence $(f_\rho)$ over $K$;
2. find minimal $P$ such that $P \in O\{Y\}$, $P(f_\rho) \rightsquigarrow 0$, and $P(f) = 0$;
3. use pseudocauchy sequence to find infinitely many zeroes of $P$ in configuration as above, contradicting the key property.
Main step

Step (3) is difficult:

**Proposition**

Suppose $K$ is asymptotic and henselian, and $k$ is linearly surjective. Let $(f_\rho)$ be a pseudocauchy sequence in $K$ and $P$ is minimal with $P(f_\rho) \rightsquigarrow 0$. Then the degree of $P$ in the cut corresponding to $(f_\rho)$ is 1.
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**Proposition**

Suppose $K$ is asymptotic and henselian, and $k$ is linearly surjective. Let $(f_\rho)$ be a pseudocauchy sequence in $K$ and $P$ is minimal with $P(f_\rho) \xrightarrow{\sim} 0$. Then the degree of $\overline{P}$ in the cut corresponding to $(f_\rho)$ is 1.

- proof is technical
- involves developing a differential newton diagram method
- problem: $\nu(f)$ does not really control $\nu(f')$
Thank you!