A topos-theoretic view of difference algebra

Ivan Tomašić

Queen Mary University of London

Workshop on Model Theory, Differential/Difference Algebra and Applications,
CUNY/Courant, 12/03/2019

Outline

Difference categories

Cohomology in difference algebra

Difference algebraic geometry

Cohomology of difference algebraic groups

Difference categories: Ritt-style

Let $\mathscr{C}$ be a category. Define its associated difference category

$$\boldsymbol{\sigma}\text{-}\mathscr{C}$$

- objects are pairs
$$(X, \sigma_X),$$
where $X \in \mathscr{C}$, $\sigma_X \in \mathscr{C}(X, X)$;
- a morphism $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a commutative diagram in $\mathscr{C}$
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f} & Y \end{array}$$
i.e., an $f \in \mathscr{C}(X, Y)$ such that
$$f \circ \sigma_X = \sigma_Y \circ f.$$

Difference categories as functor categories

Let $\boldsymbol{\sigma}$ be the category associated with the monoid $\mathbb{N}$:

- single object o;
- $\mathrm{Hom}(o, o) \simeq \mathbb{N}$.

Then

$$\boldsymbol{\sigma}\text{-}\mathscr{C} \simeq [\boldsymbol{\sigma}, \mathscr{C}],$$

the functor category:

- objects are functors $\mathcal{X} : \boldsymbol{\sigma} \to \mathscr{C}$
- morphisms are natural transformations.

Translation mechanism: if $\mathcal{X} \in [\boldsymbol{\sigma}, \mathscr{C}]$, then

$$(\mathcal{X}(o), \mathcal{X}(o \xrightarrow{1} o)) \in \boldsymbol{\sigma}\text{-}\mathscr{C}.$$

Difference categories via categorical logic

Let $\mathbb{S}$ be the algebraic theory of a single endomorphism. Then

$$\boldsymbol{\sigma}\text{-}\mathscr{C} = \mathbb{S}(\mathscr{C}),$$

the category of models of $\mathbb{S}$ in $\mathscr{C}$.

Examples

We will consider:

- $\boldsymbol{\sigma}$-$\mathbf{Set}$;
- $\boldsymbol{\sigma}$-$\mathbf{Gr}$;
- $\boldsymbol{\sigma}$-$\mathbf{Ab}$;
- $\boldsymbol{\sigma}$-$\mathbf{Rng}$.

Given $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$, consider

- R-$\mathbf{Mod}$, the category of difference R-modules.

In search of difference cohomology: path to enlightenment

Goals

- Homological algebra of $\boldsymbol{\sigma}\text{-}\mathbf{Ab}$ and $R\text{-}\mathbf{Mod}$, for $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.
- Solid foundation for difference algebraic geometry.

Awakening: obstacles to difference homological algebra

Crucial classical identities

- In $\mathbf{Set}$

$$\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

- Let $R \in \mathbf{Rng}$.
  - For $M, N \in R$-$\mathbf{Mod}$,

    $$\mathrm{Hom}_R(M, N) = R\text{-}\mathbf{Mod}(M, N)$$

    is an R-module;
  - hom-tensor duality

    $$\mathrm{Hom}_R(M \otimes N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Awakening: obstacles to difference homological algebra

Crucial classical identities fail in difference categories:

- In $\boldsymbol{\sigma}$-$\mathbf{Set}$

$$\mathrm{Hom}(X \times Y, Z) \not\simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

- Let $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$.
  - For $M, N \in R$-$\mathbf{Mod}$,

    $$\mathrm{Hom}_R(M, N) = R\text{-}\mathbf{Mod}(M, N)$$

    is a $\mathrm{Fix}(R)$-module;
  - hom-tensor duality fails

    $$\mathrm{Hom}_R(M \otimes N, P) \not\simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Insight: Monoidal closed categories

- A symmetric monoidal category $\mathscr{V}$ is closed when we have internal hom objects

$$[B, C] \in \mathscr{V}$$

so that

$$\mathscr{V}(A \otimes B, C) \simeq \mathscr{V}(A, [B, C]),$$

for all $A, B, C \in \mathscr{V}$.

- $\mathscr{V}$ is cartesian closed when monoidal closed for $\otimes = \times$.

Question

- Is $\boldsymbol{\sigma}$-$\mathbf{Set}$ cartesian closed?
- Is R-$\mathbf{Mod}$ monoidal closed (for $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$)?

Enriched categories

Let $(\mathscr{V}, \otimes, I)$ be a symmetric monoidal category.
A $\mathscr{V}$-category $\mathscr{C}$ consists of:

- a class of objects $\mathrm{Ob}(\mathscr{C})$;
- for objects X, Y in $\mathscr{C}$, a 'hom object'

$$\mathscr{C}(X, Y) \in \mathscr{V};$$

- for objects X, Y, Z, a 'composition' $\mathscr{V}$-morphism

$$\mathscr{C}(X, Y) \otimes \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z);$$

- for X in $\mathscr{C}$, a 'unit' $\mathscr{V}$-morphism

$$I \to \mathscr{C}(X, X);$$

which satisfy the expected natural conditions.

Underlying category

Let $\mathscr{V}$ be symmetric monoidal category. The points functor

$$\Gamma : \mathscr{V} \to \mathbf{Set}$$

is given by

$$\Gamma(X) = \mathscr{V}(I, X).$$

Let $\mathscr{C}$ be a $\mathscr{V}$-category. The underlying category $\mathscr{C}_0$ has:

- the same objects as $\mathscr{C}$;
- for objects X, Y,

$$\mathscr{C}_0(X, Y) = \Gamma(\mathscr{C}(X, Y)).$$

Enriched category theory

Well understood:

- enriched functor categories;
- enriched presheaves and Yoneda.

We develop:

- enriched abelian categories;
- enriched homological algebra (derived functors etc).

Essence and knowledge

Proposition

Let $(\mathscr{V}, \otimes, I)$ be a complete symmetric monoidal closed category. Then $\boldsymbol{\sigma}$-$\mathscr{V}$ is symmetric monoidal closed.

Corollary

- $\boldsymbol{\sigma}$-$\mathbf{Set}$ *is cartesian closed;*
- R-$\mathbf{Mod}$ *for* $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$ *is monoidal closed.*

What are the internal homs?

Essence and knowledge: internal homs

Consider $N = (\mathbb{N}, i \mapsto i+1) \in \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

Internal homs for $\boldsymbol{\sigma}$-Set

- $$\begin{aligned}[X,Y] &= \boldsymbol{\sigma}\text{-}\mathbf{Set}(N \times X, Y) \\ &\simeq \{(f_i) \in \mathbf{Set}(X,Y)^{\mathbb{N}} : f_{i+1} \circ \sigma_X = \sigma_Y \circ f_i\}.\end{aligned}$$
- shift $s : [X,Y] \to [X,Y]$,
$$s(f_0, f_1, \ldots) = (f_1, f_2, \ldots).$$

Internal homs for R-Mod

Given $A, B \in R\text{-}\mathbf{Mod}$,

$$[A,B]_R \in R\text{-}\mathbf{Mod}$$

is defined analogously, require $f_i \in \lfloor R \rfloor\text{-}\mathbf{Mod}(\lfloor A \rfloor, \lfloor B \rfloor)$.

Essence and knowledge

Mantra

Difference homological algebra must be developed in the framework of enriched category theory, where the relevant categories are enriched over:

- $\boldsymbol{\sigma}\text{-}\mathbf{Set}$;
- $\boldsymbol{\sigma}\text{-}\mathbf{Ab}$, (or $R\text{-}\mathbf{Mod}$ for $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$).

Note

$$\Gamma([X, Y]) = \mathrm{Fix}[X, Y] = \boldsymbol{\sigma}\text{-}\mathbf{Set}(X, Y),$$

so the Ritt-style difference algebra is the underlying category side of the enriched framework; it only sees the tip of an iceberg.

Liberation

Note

$$\boldsymbol{\sigma}\text{-}\mathbf{Set} \simeq \mathbf{B}\mathbb{N} \simeq [\boldsymbol{\sigma}, \mathbf{Set}]$$

is a Grothendieck topos (as the presheaf category on $\boldsymbol{\sigma}^{\mathsf{op}} \simeq \boldsymbol{\sigma}$), the classifying topos of $\mathbb{N}$.
Moreover,

$$\begin{aligned} \boldsymbol{\sigma}\text{-}\mathbf{Gr} &\simeq \mathbf{Gr}(\boldsymbol{\sigma}\text{-}\mathbf{Set}) \\ \boldsymbol{\sigma}\text{-}\mathbf{Ab} &\simeq \mathbf{Ab}(\boldsymbol{\sigma}\text{-}\mathbf{Set}) \\ \boldsymbol{\sigma}\text{-}\mathbf{Rng} &\simeq \mathbf{Rng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}). \end{aligned}$$

For $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$,

$$R\text{-}\mathbf{Mod} \simeq \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$$

is the category of modules in a ringed topos.

Liberation

Updated mantra

Difference algebra is the study of algebraic objects internal in the topos $\boldsymbol{\sigma}$-$\mathbf{Set}$.

Moreover:

- the above categories are categories of models of algebraic theories in $\boldsymbol{\sigma}$-$\mathbf{Set}$;
- we can apply the full power of topos theory and categorical logic;
- the enriched structure is automatic.

Topoi

A category $\mathscr{E}$ is an elementary topos if

1. $\mathscr{E}$ has finite limits (all pullbacks and a terminal object e);
2. $\mathscr{E}$ is cartesian closed;
3. $\mathscr{E}$ has a subobject classifier, i.e., an object Ω and a morphism $e \xrightarrow{t} \Omega$ such that, for each monomorphism $Y \xrightarrow{u} X$ in $\mathscr{E}$, there is a unique morphism $\chi_u : X \to \Omega$ making

$$\begin{array}{ccc} Y & \longrightarrow & e \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{\chi_u} & \Omega \end{array}$$

a pullback diagram.

Topos of difference sets

The subobject classifier in $\boldsymbol{\sigma}$-$\mathbf{Set}$ is

$$\Omega = \mathbb{N} \cup \{\infty\}, \quad \sigma_\Omega : 0 \mapsto 0, \infty \mapsto \infty, i+1 \mapsto i \ (i \in \mathbb{N}).$$

For a monomorphism $Y \xrightarrow{u} X$, the classifying map is

$$\chi_u : X \to \Omega, \ \ \chi_u(x) = \min\{n : \sigma_X^n(x) \in Y\},$$

and

$$Y = \chi_u^{-1}(\{0\}).$$

Logic of difference sets

$\Omega = \mathbb{N} \cup \{\infty\}$ is a Heyting algebra with:

- $\text{true} = 0$, $\text{false} = \infty$;
- $\wedge(i, j) = \max\{i, j\}$;
- $\vee(i, j) = \min\{i, j\}$;
- $\neg(i) = \begin{cases} 0, & i = \infty; \\ \infty, & i \in \mathbb{N}. \end{cases}$
- $\Rightarrow(i, j) = \begin{cases} 0, & i \geq j; \\ j, & i < j. \end{cases}$

Warning:

$\neg\neg \neq \mathrm{id}_\Omega$ so $\boldsymbol{\sigma}$-$\mathbf{Set}$ is not a Boolean topos.

Nirvana

Topos theory philosophy

The universe of sets can be replaced by an arbitrary base topos, and one can develop mathematics over it.

Mantra^2

Difference algebraic geometry is algebraic geometry over the base topos $\boldsymbol{\sigma}$-$\mathbf{Set}$.

Nirvana: difference schemes

M. Hakim's Zariski spectrum

For a ringed topos $(\mathscr{E}, A)$, $\mathrm{Spec.Zar}(\mathscr{E}, A)$ is the locally ringed topos equipped with a morphism of ringed topoi

$$\mathrm{Spec.Zar}(\mathscr{E}, A) \to (\mathscr{E}, A)$$

which solves a certain 2-universal problem.

Definition

The affine difference scheme associated to a difference ring A is the locally ringed topos

$$(X, \mathscr{O}_X) = \mathrm{Spec.Zar}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, A)$$

General relative schemes can be treated using stacks.

Nirvana: difference étale topos

M. Hakim's étale spectrum

For a locally ringed topos $(\mathscr{E}, A)$, $\mathrm{Spec.\acute{E}t}(\mathscr{E}, A)$ is a strictly locally ringed topos equipped with a morphism of locally ringed topoi

$$\mathrm{Spec.\acute{E}t}(\mathscr{E}, A) \to (\mathscr{E}, A)$$

which solves a certain 2-universal problem.

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme as before. Its étale topos is the strictly locally ringed topos

$$(X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}) = \mathrm{Spec.\acute{E}t}(X, \mathscr{O}_X)$$

Étale fundamental group of a difference scheme

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme, and $\bar{x} : \boldsymbol{\sigma}\text{-}\mathbf{Set} \to X_{\text{ét}}$ a point. Then

$$\pi_1^{\text{ét}}(X, \bar{x}) = \pi_1(X_{\text{ét}}, \bar{x}),$$

the Bunge-Moerdijk pro-($\boldsymbol{\sigma}\text{-}\mathbf{Set}$)-localic fundamental group associated to the geometric morphism $X_{\text{ét}} \to \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

Difference étale cohomology

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme with structure geometric morphism $\gamma : X \to \boldsymbol{\sigma}\text{-}\mathbf{Set}$, and let M be a $\mathscr{O}_{X_{\text{ét}}}$-module. Then

$$H^n_{\text{ét}}(X, M) = R^i\gamma_*(M),$$

the abelian difference groups obtained through relative (enriched) topos cohomology.

Some calculations

- (with M. Wibmer) Cohomology of difference algebraic groups. Explicit calculations for twisted groups of Lie Type as difference group schemes;
- Ext of modules over skew-polynomial rings.

Group functors

Fix a base difference ring

$$k \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}.$$

A k-difference group functor is a functor

$$\mathbf{G} : k\text{-}\mathbf{Alg} \to \mathbf{Gr}.$$

A difference algebraic group over k is a difference group functor G which is representable by a difference Hopf k-algebra A,

$$G(R) = k\text{-}\mathbf{Alg}(A, R).$$

We also consider $(\boldsymbol{\sigma}\text{-}\mathbf{Set})$-enriched k-difference group functors.

Group cohomology

Let

- $\mathbf{G}$ a k-difference group functor,
- $\mathbf{O}$ a k-difference ring functor,
- $\mathbf{F}$ a $\mathbf{G}$-$\mathbf{O}$-module.

We define

Hochschild cohomology groups

$$\mathrm{H}^n(\mathbf{G}, \mathbf{F}).$$

If $\mathbf{G}$, $\mathbf{O}$ and $\mathbf{F}$ are enriched, we define

enriched cohomology groups

$$\mathrm{H}^n[\mathbf{G}, \mathbf{F}] \in \boldsymbol{\sigma}\text{-}\mathbf{Gr}.$$

Twisted groups of Lie Type as difference group schemes

The difference group functor SU_n defined by

$$\mathrm{SU}_n(R, \sigma) = \{A \in \mathrm{SL}_n(R) : A^T \sigma(A) = I\}$$

acts on the abelian group functor su_n

$$\mathrm{su}_n(R, \sigma) = \{B \in \mathrm{sl}_n(R) : B^T + \sigma(B) = 0\}.$$

Note:

$$\mathrm{SU}_n(\bar{\mathbb{F}}_p, \mathrm{Frob}_q) = \mathrm{SU}(n, q).$$

Explicit calculations

- since SU_2 can be related to SL_2 (modulo some number theory),
$$\mathrm{H}^1(\mathrm{SU}_2, \mathrm{su}_2) = 0.$$
- in characteristic 3,
$$\mathrm{H}^1(\mathrm{SU}_3, \mathrm{su}_3) \text{ is 1-dimensional.}$$

Explicit calculations: Suzuki difference group scheme

Let $\theta : \mathrm{Sp}_4 \to \mathrm{Sp}_4$ be the algebraic endomorphism satisfying

$$\theta^2 = F_2.$$

The Suzuki difference group scheme $\mathbf{G}$:

$$\mathbf{G}(R, \sigma) = \{X \in \mathrm{Sp}_4(R) : F_2 \circ \sigma(X) = \theta(X)\}.$$

naturally acts on the module

$$\mathbf{F}(R, \sigma) = \{(x_1, x_2, x_3, x_4)^T \in R^4 : \sigma^2 x_i^2 = x_i\}.$$

Note

$$\mathbf{G}(\bar{\mathbb{F}}_2, F_q) = {}^2B_2(2q^2),$$

the (familiar) finite Suzuki group.
We have

$$\mathrm{H}^1(\mathbf{G}, \mathbf{F}) \text{ is 1-dimensional.}$$

Extensions of modules over skew-polynomial rings

For $k \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, have the skew-polynomial ring

$$R = k[T; \sigma_k].$$

Equivalence of categories:

$$k\text{-}\mathbf{Mod} \simeq R\text{-}\mathbf{Mod}.$$

If F is an étale k-module, then

$$\mathrm{Ext}^i_{R\text{-}\mathbf{Mod}}(F, F') = \begin{cases} [F, F']_s, & i = 1, \\ 0, & i > 1, \end{cases}$$

where $[F, F']_s = [F, F']/\operatorname{Im}(s - \mathrm{id})$ is the module of s-coinvariants of $[F, F']$.
In particular, if k is linearly difference closed and F, F' are finite étale, then, for $i > 0$,

$$\mathrm{Ext}^i(F, F') = 0.$$

Studying Elephant