A topos-theoretic view of difference algebra

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Workshop on Model Theory, Differential/Difference Algebra and Applications, CUNY/Courant, 12/03/2019
Outline

Difference categories

Cohomology in difference algebra

Difference algebraic geometry

Cohomology of difference algebraic groups
Difference categories: Ritt-style

Let $\mathcal{C}$ be a category. Define its associated difference category $\sigma\mathcal{C}$

- **objects** are pairs $(X, \sigma_X)$,

  where $X \in \mathcal{C}$, $\sigma_X \in \mathcal{C}(X, X)$;

- **a morphism** $f : (X, \sigma_X) \rightarrow (Y, \sigma_Y)$ is a commutative diagram in $\mathcal{C}$

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\sigma_X} & & \downarrow{\sigma_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

i.e., an $f \in \mathcal{C}(X, Y)$ such that

\[f \circ \sigma_X = \sigma_Y \circ f.\]
Let \( \sigma \) be the category associated with the monoid \( \mathbb{N} \):

- single object \( o \);
- \( \text{Hom}(o, o) \cong \mathbb{N} \).

Then

\[ \sigma \dashv \mathcal{C} \cong [\sigma, \mathcal{C}] , \]

the functor category:

- objects are functors \( \mathcal{X} : \sigma \to \mathcal{C} \)
- morphisms are natural transformations.

Translation mechanism: if \( \mathcal{X} \in [\sigma, \mathcal{C}] \), then

\[ (\mathcal{X}(o), \mathcal{X}(o \overset{1}{\rightarrow} o)) \in \sigma \dashv \mathcal{C}. \]
Let $\mathcal{S}$ be the algebraic theory of a single endomorphism. Then

$$\sigma^{-}\mathcal{C} = \mathcal{S}(\mathcal{C}),$$

the category of models of $\mathcal{S}$ in $\mathcal{C}$. 
Examples

We will consider:

- $\sigma$-Set;
- $\sigma$-Gr;
- $\sigma$-Ab;
- $\sigma$-Rng.

Given $R \in \sigma$-Rng, consider

- $R$-Mod, the category of difference $R$-modules.
In search of difference cohomology: path to enlightenment

Goals

▶ Homological algebra of $\sigma$-Ab and $R$-Mod, for $R \in \sigma$-Rng.
▶ Solid foundation for difference algebraic geometry.
Awakening: obstacles to difference homological algebra

Crucial classical identities

- In $\text{Set}$

\[ \text{Hom}(X \times Y, Z) \simeq \text{Hom}(X, \text{Hom}(Y, Z)). \]

- Let $R \in \text{Rng}$.
  - For $M, N \in R\text{-Mod}$,
    \[ \text{Hom}_R(M, N) = R\text{-Mod}(M, N) \]
    is an $R$-module;
  - hom-tensor duality
    \[ \text{Hom}_R(M \otimes N, P) \simeq \text{Hom}_R(M, \text{Hom}_R(N, P)). \]
Awakening: obstacles to difference homological algebra

Crucial classical identities fail in difference categories:

- In $\sigma$-$\text{Set}$
  \[ \text{Hom}(X \times Y, Z) \not\cong \text{Hom}(X, \text{Hom}(Y, Z)). \]

- Let $R \in \sigma$-$\text{Rng}$.
  - For $M, N \in R$-$\text{Mod}$,
    \[ \text{Hom}_R(M, N) = R$-$\text{Mod}(M, N) \]
    is a $\text{Fix}(R)$-module;
  - hom-tensor duality fails
    \[ \text{Hom}_R(M \otimes N, P) \not\cong \text{Hom}_R(M, \text{Hom}_R(N, P)). \]
Insight: Monoidal closed categories

- A symmetric monoidal category $\mathcal{V}$ is closed when we have internal hom objects

\[ [B, C] \in \mathcal{V} \]

so that

\[ \mathcal{V}(A \otimes B, C) \simeq \mathcal{V}(A, [B, C]), \]

for all $A, B, C \in \mathcal{V}$.

- $\mathcal{V}$ is cartesian closed when monoidal closed for $\otimes = \times$.

Question

- Is $\sigma$-Set cartesian closed?
- Is $R$-Mod monoidal closed (for $R \in \sigma$-$\text{Rng}$)?
Enriched categories

Let \((\mathcal{V}, \otimes, I)\) be a symmetric monoidal category. A \(\mathcal{V}\)-category \(\mathcal{C}\) consists of:

- a class of objects \(\text{Ob}(\mathcal{C})\);
- for objects \(X, Y\) in \(\mathcal{C}\), a ‘hom object’ \(\mathcal{C}(X, Y) \in \mathcal{V}\);
- for objects \(X, Y, Z\), a ‘composition’ \(\mathcal{V}\)-morphism \(\mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)\);
- for \(X\) in \(\mathcal{C}\), a ‘unit’ \(\mathcal{V}\)-morphism \(I \rightarrow \mathcal{C}(X, X)\);

which satisfy the expected natural conditions.
Let $\mathcal{V}$ be symmetric monoidal category. The points functor

$$\Gamma : \mathcal{V} \to \text{Set}$$

is given by

$$\Gamma(X) = \mathcal{V}(I, X).$$

Let $\mathcal{C}$ be a $\mathcal{V}$-category. The underlying category $\mathcal{C}_0$ has:

- the same objects as $\mathcal{C}$;
- for objects $X, Y$,

$$\mathcal{C}_0(X, Y) = \Gamma(\mathcal{C}(X, Y)).$$
Enriched category theory

Well understood:
- enriched functor categories;
- enriched presheaves and Yoneda.

We develop:
- enriched abelian categories;
- enriched homological algebra (derived functors etc).
Proposition

Let $(\mathcal{V}, \otimes, I)$ be a complete symmetric monoidal closed category. Then $\sigma^{-}\mathcal{V}$ is symmetric monoidal closed.

Corollary

- $\sigma^{-}\text{Set}$ is cartesian closed;
- $\mathcal{R}\text{-Mod}$ for $\mathcal{R} \in \sigma^{-}\text{Rng}$ is monoidal closed.

What are the internal homs?
Consider \( N = (\mathbb{N}, i \mapsto i + 1) \in \sigma\text{-Set} \).

**Internal homs for \( \sigma\text{-Set} \)**

\[
[X, Y] = \sigma\text{-Set}(N \times X, Y)
\]
\[
\simeq \{(f_i) \in \text{Set}(X, Y)^\mathbb{N} : f_{i+1} \circ \sigma_X = \sigma_Y \circ f_i\}.
\]

**Shift \( s : [X, Y] \to [X, Y] \),**

\[
s(f_0, f_1, \ldots) = (f_1, f_2, \ldots).
\]

**Internal homs for \( R\text{-Mod} \)**

Given \( A, B \in R\text{-Mod} \),

\[
[A, B]_R \in R\text{-Mod}
\]

is defined analogously, require \( f_i \in [R]\text{-Mod}(\langle A \rangle, \langle B \rangle) \).
## Essence and knowledge

### Mantra

Difference homological algebra must be developed in the framework of **enriched category theory**, where the relevant categories are enriched over:

- $\sigma\text{-Set}$
- $\sigma\text{-Ab}$, (or $R\text{-Mod}$ for $R \in \sigma\text{-Rng}$).

**Note**

$$\Gamma([X,Y]) = \text{Fix}[X,Y] = \sigma\text{-Set}(X,Y),$$

so the Ritt-style difference algebra is the **underlying category side** of the enriched framework; it only sees the tip of an iceberg.
Note

\[ \sigma\text{-Set} \cong B\mathbb{N} \cong [\sigma, \text{Set}] \]

is a Grothendieck topos (as the presheaf category on \( \sigma^{\text{op}} \cong \sigma \)), the classifying topos of \( \mathbb{N} \).
Moreover,

\[ \sigma\text{-Gr} \cong \text{Gr}(\sigma\text{-Set}) \]
\[ \sigma\text{-Ab} \cong \text{Ab}(\sigma\text{-Set}) \]
\[ \sigma\text{-Rng} \cong \text{Rng}(\sigma\text{-Set}) \]

For \( R \in \sigma\text{-Rng} \),

\[ R\text{-Mod} \cong \text{Mod}(\sigma\text{-Set}, R) \]

is the category of modules in a ringed topos.
Updated mantra

Difference algebra is the study of algebraic objects \textit{internal} in the topos $\sigma\text{-Set}$.

Moreover:

- the above categories are categories of models of \textit{algebraic theories} in $\sigma\text{-Set}$;
- we can apply the full power of \textit{topos theory} and \textit{categorical logic};
- the \textit{enriched structure} is automatic.
A category $\mathcal{E}$ is an elementary topos if

1. $\mathcal{E}$ has finite limits (all pullbacks and a terminal object $e$);
2. $\mathcal{E}$ is cartesian closed;
3. $\mathcal{E}$ has a subobject classifier, i.e., an object $\Omega$ and a morphism $e \xrightarrow{t} \Omega$ such that, for each monomorphism $Y \xrightarrow{u} X$ in $\mathcal{E}$, there is a unique morphism $\chi_u : X \rightarrow \Omega$ making

$$
\begin{array}{ccc}
Y & \longrightarrow & e \\
\downarrow u & & \downarrow t \\
X & \xrightarrow{\chi_u} & \Omega
\end{array}
$$

a pullback diagram.
Topos of difference sets

The subobject classifier in $\sigma$-Set is

$$\Omega = \mathbb{N} \cup \{\infty\}, \quad \sigma_\Omega : 0 \mapsto 0, \infty \mapsto \infty, \ i + 1 \mapsto i \ (i \in \mathbb{N}).$$

For a monomorphism $Y \overset{u}{\to} X$, the classifying map is

$$\chi_u : X \to \Omega, \quad \chi_u(x) = \min\{n : \sigma^n_X(x) \in Y\},$$

and

$$Y = \chi_u^{-1}(\{0\}).$$
Logic of difference sets

\[ \Omega = \mathbb{N} \cup \{\infty\} \] is a Heyting algebra with:

- true = 0, false = \infty;
- \( \land (i, j) = \max\{i, j\} \);
- \( \lor (i, j) = \min\{i, j\} \);
- \( \neg(i) = \begin{cases} 0, & i = \infty; \\ \infty, & i \in \mathbb{N}. \end{cases} \)
- \( \Rightarrow(i, j) = \begin{cases} 0, & i \geq j; \\ j, & i < j. \end{cases} \)

Warning:

\( \neg \neg \neq \text{id}_\Omega \) so \( \sigma\text{-Set} \) is not a Boolean topos.
Topos theory philosophy

The universe of sets can be replaced by an arbitrary base topos, and one can develop mathematics over it.

Mantra\(^2\)

Difference algebraic geometry is algebraic geometry over the base topos \(\sigma\text{-Set}\).
M. Hakim’s Zariski spectrum

For a ringed topos \((\mathcal{E}, A)\), \(\text{Spec.Zar}(\mathcal{E}, A)\) is the locally ringed topos equipped with a morphism of ringed topoi

\[
\text{Spec.Zar}(\mathcal{E}, A) \rightarrow (\mathcal{E}, A)
\]

which solves a certain 2-universal problem.

Definition

The affine difference scheme associated to a difference ring \(A\) is the locally ringed topos

\[
(X, \mathcal{O}_X) = \text{Spec.Zar}(\sigma\text{-Set}, A)
\]

General relative schemes can be treated using stacks.
M. Hakim’s étale spectrum

For a locally ringed topos \((\mathcal{E}, A)\), \(\text{Spec.Ét}(\mathcal{E}, A)\) is a strictly locally ringed topos equipped with a morphism of locally ringed topoi

\[
\text{Spec.Ét}(\mathcal{E}, A) \to (\mathcal{E}, A)
\]

which solves a certain 2-universal problem.

Definition

Let \((X, \mathcal{O}_X)\) be a difference scheme as before. Its étale topos is the strictly locally ringed topos

\[
(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}) = \text{Spec.Ét}(X, \mathcal{O}_X)
\]
Étale fundamental group of a difference scheme

**Definition**

Let \((X, \mathcal{O}_X)\) be a difference scheme, and \(\bar{x} : \sigma\text{-Set} \to X_{\text{ét}}\) a point. Then

\[
\pi_{\text{ét}}^1(X, \bar{x}) = \pi_1(X_{\text{ét}}, \bar{x}),
\]

the Bunge-Moerdijk pro-(\(\sigma\text{-Set}\))-localic fundamental group associated to the geometric morphism \(X_{\text{ét}} \to \sigma\text{-Set}\).
Difference étale cohomology

Definition

Let $(X, \mathcal{O}_X)$ be a difference scheme with structure geometric morphism $\gamma : X \to \sigma\text{-}\text{Set}$, and let $M$ be a $\mathcal{O}_{X_{\text{ét}}}$-module. Then

$$H_{\text{ét}}^n(X, M) = R^i\gamma_*(M),$$

the abelian difference groups obtained through relative (enriched) topos cohomology.
Some calculations

- (with M. Wibmer) Cohomology of difference algebraic groups. Explicit calculations for twisted groups of Lie Type as difference group schemes;
- Ext of modules over skew-polynomial rings.
Group functors

Fix a base difference ring

\[ k \in \sigma\text{-Rng}. \]

A \( k \)-difference group functor is a functor

\[ G : k\text{-Alg} \rightarrow \text{Gr}. \]

A difference algebraic group over \( k \) is a difference group functor \( G \) which is representable by a difference Hopf \( k \)-algebra \( A \),

\[ G(R) = k\text{-Alg}(A, R). \]

We also consider \( (\sigma\text{-Set}) \)-enriched \( k \)-difference group functors.
Group cohomology

Let

- $G$ a $k$-difference group functor,
- $O$ a $k$-difference ring functor,
- $F$ a $G$-$O$-module.

We define

Hochschild cohomology groups

$$H^n(G, F).$$

If $G$, $O$ and $F$ are enriched, we define

enriched cohomology groups

$$H^n[G, F] \in \sigma\text{-Gr}.$$
Twisted groups of Lie Type as difference group schemes

The difference group functor $SU_n$ defined by

$$SU_n(R, \sigma) = \{ A \in SL_n(R) : A^T \sigma(A) = I \}$$

acts on the abelian group functor $su_n$

$$su_n(R, \sigma) = \{ B \in sl_n(R) : B^T + \sigma(B) = 0 \}.$$

Note:

$$SU_n(\overline{F}_p, \text{Frob}_q) = SU(n,q).$$
since $SU_2$ can be related to $SL_2$ (modulo some number theory),

$$H^1(SU_2, su_2) = 0.$$ 

in characteristic 3,

$$H^1(SU_3, su_3)$$ is 1-dimensional.
Explicit calculations: Suzuki difference group scheme

Let \( \theta : \Sp_4 \rightarrow \Sp_4 \) be the algebraic endomorphism satisfying

\[
\theta^2 = F_2.
\]

The Suzuki difference group scheme \( G \):

\[
G(R, \sigma) = \{ X \in \Sp_4(R) : F_2 \circ \sigma(X) = \theta(X) \}.
\]

naturally acts on the module

\[
F(R, \sigma) = \{(x_1, x_2, x_3, x_4)^T \in R^4 : \sigma^2 x_i^2 = x_i \}.
\]

Note

\[
G(\overline{F}_2, F_q) = 2B_2(2q^2),
\]

the (familiar) finite Suzuki group.

We have

\[
H^1(G, F) \text{ is 1-dimensional.}
\]
Extensions of modules over skew-polynomial rings

For \( k \in \sigma\text{-Rng} \), have the skew-polynomial ring

\[
R = k[T; \sigma_k].
\]

Equivalence of categories:

\[
k\text{-Mod} \cong R\text{-Mod}.
\]

If \( F \) is an étale \( k \)-module, then

\[
\text{Ext}^i_{R\text{-Mod}}(F, F') = \begin{cases} 
[F, F']_s, & i = 1, \\
0, & i > 1,
\end{cases}
\]

where \([F, F']_s = [F, F']/\text{Im}(s - \text{id})\) is the module of \(s\)-coinvariants of \([F, F']\).

In particular, if \( k \) is linearly difference closed and \( F, F' \) are finite étale, then, for \( i > 0 \),

\[
\text{Ext}^i(F, F') = 0.
\]
Studying Elephant