Effective Integration of Polynomial Differential Equations

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Kolchin Seminar 2019
The Problem

Given a polynomial differential equation

\[ F(x,y,y',y'',y''',...) = 0 \]

determine (algorithmically) if all its solutions are algebraic.

We are mainly interested on ordinary equations linear on \( y' \) (first order and of first degree).

Pencil sketches of phase flows of vector fields by Eugene Zhang
( http://web.engr.oregonstate.edu/~zhange/vecfld_design.html)
History
\[ (A.) \quad \frac{d^2 y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \frac{dy}{dx} - \frac{\alpha \cdot \beta}{x(1-x)} \cdot y = 0 \]

Gauss hypergeometric equation
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Ueber diejenigen Fälle, in welchen die *Gaussische hypergeometrische Reihe* eine *algebraische* Function ihres vierten Elementes darstellt.

Nebst zwei Figurentafeln.

(Von Herrn H. A. Schwarz in Zürich.)

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Gauss hypergeometric equation

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Schwarz's List

List of parameters for which all solutions of Gauss hypergeometric equation are algebraic.
Why Schwarz succeeded?

A Gauss hypergeometric equation is a linear differential equation over the Riemann sphere with 3 poles.

Its monodromy is a representation of the punctured sphere in $GL(2)$.

If all the solutions are algebraic then the local monodromies are of finite order.
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H. A. Schwarz [15] determined all such operators with three singular points whose kernel consists of algebraic functions. His method was to show that if $B$ and $C$ lie in $\mathbb{R}(x)$ then the monodromy group can be calculated from the group generated by the reflections relative to three circles which meet at angles determined by the exponent differences of $L$. He used this to show that the solutions of $L$ are all algebraic if and only if these angles coincide with the angles of a spherical triangle whose vertices are fixed points of three rotations any pair of which generates a finite rotation group.
Generalizations

A lot of activity on the problem followed in the course of subsequent years. There are contributions on the subject by Fuchs, Jordan, Poincaré, Painlevé, Boulanger, Pépin, Frobenius, Halphen and others.

In particular, motivated by this problem, Jordan proved that any finite subgroup of $GL(n)$ has an abelian normal subgroup of index bounded by a computable function $J(n)$. 
Felix Klein at the beginning of Section 3 of Chapter V of his Lectures on the Icosahedron writes:

"(...) we now concern ourselves (...) with the problem: to present all linear homogenous differential equations of the second order with rational coefficients:

\[ y'' + p y' + q y = 0 \]

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By the end of XIXth century the problem for linear differential equations was considered completely solved.
What's next?

Liste des attributions du Grand Prix des Sciences Mathématiques depuis 1881 jusqu’à 1915.

1890. Sujet proposé. — Perfectionner en un point important la théorie des équations différentielles du premier ordre et du premier degré.
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What's next?

SUR L'INTÉGRATION ALGÉBRIQUE DES ÉQUATIONS DIFFÉRENTIELLES DU PREMIER ORDRE ET DU PREMIER DEGRÉ;
par M. H. Poincaré, à Paris.

Adresses du 26 avril 1892.

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Pour reconnaître si une équation différentielle du 1er ordre et du 1er degré est intégrable algébriquement, il suffit évidemment de trouver une limite supérieure du degré de l’intégrale; il ne reste plus ensuite qu’â effectuer des calculs purement algébriques.

A result by Poincaré

Assuming that all singularities are either centers or radials singularities, Poincaré provided an explicit bound for the degree of the general algebraic leaf.
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**Limitation du degré.**

Dans le cas où tous les cols sont du 1er ou du 2e genre, il est possible de trouver une limite supérieure du degré \( p \) et par conséquent de reconnaître si l'équation est intégrable algébriquement.

Nous venons de trouver, en effet, sans avoir besoin de supposer que tous les nœuds soient dicritiques:

\[(m + 2) = p\left(\frac{1}{\alpha_i} + \frac{1}{\alpha_s}\right),\]

d'où:

\[\alpha_i\alpha_s(m + 2) = p(\alpha_i + \alpha_s).\]

Or, \( \alpha_i \) et \( \alpha_s \) sont premiers entre eux et par conséquent chacun d'eux est premier avec \( \alpha_i + \alpha_s \). Donc \( \alpha_i + \alpha_s \) divise \( m + 2 \).

Nous devons en conclure que \( \alpha_i + \alpha_s \) et par conséquent \( \alpha_i, \alpha_s \) et \( p \) sont limités. c. q. f. d.

Je m'arrêterai là, bien que les principes qui précèdent puissent probablement, avec de légères modifications, donner des résultats dans des cas moins particuliers.
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**Poincaré Problem [after Cerveau-Lins Neto]**
Bound the degree of algebraic curves invariant by foliations of the projective plane in function of the degree of the foliation and combinatorial data attached to the singularities of the foliation.
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There exist one parameter families of foliations of fixed degree and fixed analytical type of singularities such that for a dense set of parameters the foliations are by algebraic leaves and the degree is unbounded.
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Later shown by McQuillan to be quotients of linear flows on abelian surfaces
Revisiting the linear case

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They show that in order to decide if all the solutions of linear differential equation of arbitrary rank are algebraic it suffices to be able to decide if the solutions of a rank one differential equation with algebraic coefficients

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The general questions of this nature which arise in connection with integrals of the form

\[ \int \frac{Q}{\sqrt{X}} \, dx, \]

or, more generally,

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are of extreme interest and difficulty. The case which has received most attention is that in which \( m = 2 \) and \( X \) is of the third or fourth degree, in which case the integral is said to be \textit{elliptic}. An integral of this kind is called \textit{pseudo-elliptic} if it is expressible in terms of algebraical and logarithmic functions.

But no method has been devised as yet by which we can always determine in a finite number of steps whether a \textit{given} elliptic integral is pseudo-elliptic, and integrate it if it is. and there is reason to suppose that no such method can be given.
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THE SOLUTION OF THE PROBLEM OF INTEGRATION
INFINITE TERMS

BY ROBERT H. RISCH

Communicated by M. H. Protter, October 22, 1969
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In the above special case of Abelian integrals, the fact that integration reduces to divisor testing was first explicitly (albeit, somewhat obliquely) stated by Goursat in 1894 [5, p. 516]. At that time the problem was considered exceedingly difficult or even undecidable (before Gödel!). See the remarks of Halphen [6, last page], Goursat [5, p. 516], and Hardy [7, pp. 8–11, 47–48, 52]. The only criteria they considered was the highly nonconstructive one given by Abel’s theorem.
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A bound for the torsion. It has been conjectured that there is a universal bound, depending only on the genus and the ground field for torsion on the Jacobian variety of an algebraic curve defined over a finitely generated field $k$. (See [2, p. 264] for some discussion of the elliptic curve case.) When integrating a given elementary function, one needs only to be able to find the bound for an explicitly given curve. This can be done using things now in the repertory of arithmetical algebraic geometry. One method is outlined below.

Risch showed how to (algorithmically) decide whether the solutions of $y' = a(x)y$ are algebraic or not by considering reduction to positive characteristic of the equations involved.

This completes the solution of our problem for linear differential equations.
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New Developments
Effective algebraic integration

**Conjecture 1.1.** The Zariski closure in $\mathbb{P}H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(d - 1))$ of the set of foliations of degree $d$ on $\mathbb{P}^2$ which admit a rational integral consists in transversely projective foliations.
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- Algebraically integrable foliations
- Transversely euclidean foliations
- Transversely affine foliations
- Transversely projective foliations

- Rational first integrals
- Closed rational 1-forms
- Existence of first integral on a Liouvillian extension
- Existence of a first integral on a Picard-Vessiot extension

- (log)-primitives of closed differential forms
- Solutions of linear differential equations
Effective algebraic integration

Conjecture 1.1. The Zariski closure in $\mathbb{P}H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(d - 1))$ of the set of foliations of degree $d$ on $\mathbb{P}^2$ which admit a rational integral consists in transversely projective foliations.
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TOWARD EFFECTIVE LIOUVILLIAN INTEGRATION

GAËL COUSIN, ALCIDES LINS NETO, AND JORGE VITÓRIO PEREIRA
Theorem A. Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on the projective plane $\mathbb{P}^2$. Assume that $\mathcal{F}$ admits a Liouvillian first integral but does not admit a rational first integral. Then $\mathcal{F}$ admits an algebraic invariant curve of degree at most $12(d-1)$. 
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- Existence of pluricanonical section using logarithmic symmetric differentials
- Structure of transversely affine foliations (Cousin,P.)
- Finite cyclic quotients of closed rational 1-forms
- Effective generation of the pluricanonical systems
- Rational pull-backs of Riccati foliations
- Transversely affine foliations
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- Effective non-vanishing of adjoint linear series (Demailly, Kóllar, Ein-Lazarsfeld)
- Bound on the multiplicities of irreducible components of relatively minimal hyperbolic fibrations
- Zariski decomposition of the canonical bundle of a relatively minimal foliation (McQuillan)
Theorem C. The Zariski closure in $\mathbb{P}H^0(\mathbb{P}^2, T\mathbb{P}^2(d-1))$ of the set of degree $d$ foliations admitting a rational first integral with general fiber of genus $\leq g$ is formed by transversely projective foliations.
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Classification of foliations in terms of Kodaira dimension (McQuillan, Brunella, Mendes)

Analogue classification in terms of adjoint dimension (better behaved in families)

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### Table 1. Classification of foliations according to their adjoint/Kodaira dimensions.

<table>
<thead>
<tr>
<th>adj</th>
<th>kod</th>
<th>Description</th>
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<tr>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>Rational fibration</td>
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<td>0</td>
<td>0</td>
<td>Finite quotient of Riccati foliation generated by global vector field</td>
</tr>
<tr>
<td>1</td>
<td>Riccati foliation</td>
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<tr>
<td>0</td>
<td>0</td>
<td>Finite quotient of linear foliation on a torus</td>
</tr>
<tr>
<td>1</td>
<td>Finite quotient of $E \times C \to C$, $g(C) \geq 2$</td>
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</tr>
<tr>
<td>1</td>
<td>Finite quotient of $E \times C \to E$, $g(C) \geq 2$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Turbulent foliation</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Non-isotrivial elliptic fibration</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$-\infty$</td>
<td>Irreducible quotient of $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$</td>
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<tr>
<td>1</td>
<td>Finite quotient of $C_1 \times C_2 \to C_1$, $g(C_i) \geq 2$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>General type</td>
<td></td>
</tr>
</tbody>
</table>

Analogue classification in terms of adjoint dimension (better behaved in families)

Classification of foliations in terms of Kodaira dimension (McQuillan, Brunella, Mendes)