Example 2 of Galois Connection

- If
  - $@ \in \mathcal{D}^1 \longmapsto \mathcal{D}^2$
  - $\alpha \in \wp(\mathcal{D}^1) \longmapsto \wp(\mathcal{D}^2)$ direct image (2)
    $\alpha(X) \stackrel{\text{def}}{=} \{@(x) \mid x \in X\}$
  - $\gamma \in \wp(\mathcal{D}^2) \longmapsto \wp(\mathcal{D}^1)$ inverse image
    $\gamma(Y) \stackrel{\text{def}}{=} \{x \mid @(x) \in Y\}$

  then

$$\langle \wp(\mathcal{D}^1), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \wp(\mathcal{D}^2), \subseteq \rangle \qquad (3)$$

(see Ex. 5.5.2).

Duality Principle

- We write $\leq^{-1}$ or $\geq$ for the inverse of the partial order $\leq$.
- Observe that:

$$\langle \mathcal{D}^1, \sqsubseteq^1 \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathcal{D}^2, \sqsubseteq^2 \rangle$$

  if and only if

$$\langle \mathcal{D}^2, \sqsupseteq^2 \rangle \underset{\gamma}{\overset{\alpha}{\leftrightarrows}} \langle \mathcal{D}^1, \sqsupseteq^1 \rangle$$

- It follows that the duality principle on posets stating that any theorem is true for all posets, then so is its dual obtained by substituting $\geq$, $>$, $\top$, $\bot$, $\vee$, $\wedge$, etc. respectively for $\leq$, $<$, $\bot$, $\top$, $\wedge$, $\vee$, etc. can be extended to Galois connections by exchanging α and γ.

Moore Families

- A *Moore family* is a subset of a complete lattice containing $\top^1$ and closed under arbitrary glbs $\sqcap^1$;
- If $\langle \mathcal{D}^1, \sqsubseteq^1 \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathcal{D}^2, \sqsubseteq^2 \rangle$ and $\mathcal{D}^1(\sqsubseteq^1, \bot^1, \top^1, \sqcap^1, \sqcup^1)$ is a complete lattice then $\gamma(\mathcal{D}^2)$ is a Moore family (19)
(see Ex. 5.5.19).
- A consequence is that one can reason upon the abstract semantics using only $\mathcal{D}^1$ and the image of $\mathcal{D}^1$ by the closure operator $\gamma \circ \alpha$ (instead of $\mathcal{D}^2$).

One Adjoint Function Determines the Other

- If $\langle \mathcal{D}^1, \sqsubseteq^1 \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathcal{D}^2, \sqsubseteq^2 \rangle$, then:

$$\begin{aligned} \forall x \in \mathcal{D}^1 : \alpha(x) &= \sqcap^2 \{ y \mid x \sqsubseteq^1 \gamma(y) \} \\ \forall y \in \mathcal{D}^2 : \gamma(y) &= \sqcup^1 \{ x \mid \alpha(x) \sqsubseteq^2 y \} \end{aligned} \qquad (20)$$

(see Ex. 5.5.21).

Preservation of Lubs/Glbs

If $\langle \mathcal{D}^1, \sqsubseteq^1 \rangle \xrightleftharpoons[\alpha]{\gamma} \langle \mathcal{D}^2, \sqsubseteq^2 \rangle$, then:

- α preserves existing lubs: if $\sqcup^1 X$ exists, then $\alpha\left(\sqcup^1 X\right)$ is the lub of $\{\alpha(x) \mid x \in X\}$;
- γ preserves existing glbs: if $Y \subseteq \mathcal{D}^2$ and $\sqcap^2 Y$ exists, then $\gamma\left(\sqcap^2 Y\right)$ is the glb of $\{\gamma(y) \mid y \in Y\}$.

(see Ex. 5.5.22).

5.6.2 Answer to Exercise 2

Proof $\alpha(X) \subseteq Y \Leftrightarrow \{@(x) \mid x \in X\} \subseteq Y \Leftrightarrow \forall x \in X : @(x) \in Y$ $\Leftrightarrow X \subseteq \{x \mid @(x) \in Y\} \Leftrightarrow X \subseteq \gamma(Y)$. □

5.6.19 Answer to Exercise 19

Proof

- If $x \in \gamma(\mathcal{D}^2)$ then $\exists y \in \mathcal{D}^2 : x = \gamma(y) \sqsubseteq^1 \top^1$. By monotony $\gamma \circ \alpha \circ \gamma(y) \sqsubseteq^1 \gamma \circ \alpha(\top^1) = \top^1$ since $\gamma \circ \alpha$ is extensive, $\top^1$ is the supremum and $\sqsubseteq^1$ is antisymetric. We have $\gamma \circ \alpha \circ \gamma(y)$ $= \gamma(y)$ so $x \sqsubseteq^1 \top^1$, proving that $\top^1 \in \gamma(\mathcal{D}^2)$ is the supremum of $\gamma(\mathcal{D}^2)$;

- Assume that $X \subseteq \gamma(\mathcal{D}^2)$. If $x \in X$, then $\exists y \in \mathcal{D}^2$ such that $x = \gamma(y)$. Then $\sqcap^1 X$ exists in a complete lattice and satisfies $\sqcap^1 X \sqsubseteq^1 x$ so that by monotony and $\gamma \circ \alpha \circ \gamma = \gamma$, $\gamma \circ \alpha(\sqcap^1 X) \sqsubseteq^1 \gamma \circ \alpha(x) = \gamma \circ \alpha \circ \gamma(y) = \gamma(y) = x$ proving that $\gamma \circ \alpha(\sqcap^1 X)$ is a lower bound of X so that $\gamma \circ \alpha(\sqcap^1 X) \sqsubseteq^1 \sqcap^1 X$. But $\gamma \circ \alpha$ is extensive so that by antisymmetry $\gamma \circ \alpha(\sqcap^1 X) = \sqcap^1 X$ proving that $\sqcap^1 X \in \gamma(\mathcal{D}^2)$.

□

5.6.21 Answer to Exercise 21

Proof If $x \sqsubseteq^1 \gamma(y)$ then $\alpha(x) \sqsubseteq^2 y$ by definition of a Galois connection so that $\alpha(x)$ is a lower bound of $\{y \mid x \sqsubseteq^1 \gamma(y)\}$. Moreover $x \sqsubseteq^1 \gamma \circ \alpha(x)$ since $\gamma \circ \alpha$ is extensive so that $\alpha(x)$ belongs to $\{y \mid x \sqsubseteq^1 \gamma(y)\}$. It follows that $\alpha(x)$ is the greatest lower bound of $\{y \mid x \sqsubseteq^1 \gamma(y)\}$ since for any other lower bound ℓ, we must have $\ell \sqsubseteq^2 \alpha(x)$.

The dual result holds for γ. □

5.6.22 Answer to Exercise 22

- α preserves existing lubs:

  Proof Assume that X is a subset of $\mathcal{D}^1$ such that $\sqcup^1 X$ exists. For all $x \in X$ we have $x \sqsubseteq^1 \sqcup^1 X$ by definition of lubs so that $\alpha(x) \sqsubseteq^2 \alpha(\sqcup^1 X)$ by monotony, proving that $\alpha(\sqcup^1 X)$ is an upper bound of the $\alpha(x)$.

  Let m be another upper bound of all $\alpha(x)$, $x \in X$. We have $\alpha(x) \sqsubseteq^2 m$, whence $x \sqsubseteq^1 \gamma(m)$ by definition of a Galois connection so that $\sqcup^1 X \sqsubseteq^1 \gamma(m)$ by definition of lubs. By monotony and $\alpha \circ \gamma$ is reductive it follows that $\alpha(\sqcup^1 X) \sqsubseteq^2 \alpha \circ \gamma(m) \sqsubseteq^2 m$, proving that $\alpha(\sqcup^1 X)$ is the lub of $\{\alpha(x) \mid x \in X\}$.
  $\square$

- γ preserves existing glbs:

  Proof By the duality principle. □