Design of Semantics by Abstract Interpretation

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[1] Extended version of the invited address at MFPS XIII, CMU, Pittsburgh, March 24, 1997

Content

Application of abstract interpretation ideas to the design of formal semantics:

- Examples of abstract interpretations;
- Abstraction of fixpoint semantics;
  - Maximal **trace semantics** of nondeterministic transition systems;
  - Abstraction of the trace into a natural/demoniac/angelic **relational semantics**;

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.../...

  - Abstraction of the relational into a **nondeterministic** Plotkin/-Smyth/Hoare **denotational/functional semantics**;
  - Abstraction of the natural/demoniac relational into a **deterministic denotational/functional semantics**; Scott's semantics;
  - Abstraction of nondeterministic denotational semantics to weakest precondition/strongest postcondition **predicate transformer semantics**;
  - Abstraction of predicate transformer semantics to à la Hoare **axiomatic semantics**; Program proof methods;
- Extension to the λ-calculus.

Examples of Abstract Interpretations

Applications of Abstract Interpretation

- Mainly used for specifying *program analyzers* constructively derived from a formal semantics;
- Such analyzers can be used to statically and fully automatically determine *run-time properties of programs*;
- Such run-time information can be used in complement to classical program provers, model-checkers, ... for program *verification* (abstract debugging, ...) and *transformation* (compiler optimization, partial evaluation, parallelization, ...);
- We will show that abstract interpretation can be used to relate and *design program semantics* (and program proof methods).

Approximation

- The central idea of abstract interpretation [2, 3] is that of *approximation*;
- A *program analyzer* computes a finite approximation of the infinite set of possible run-time behaviors of the program for all possible execution environments (inputs, interrupts, ...);
- A *program semantics* specifies an approximation of the run-time program behaviors in all possible execution environments abstracting away from implementation details.

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[2] P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Conference Record of the Fourth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 238–252, Los Angeles, California, 1977. ACM Press, New York, New York, USA.

[3] P. Cousot and R. Cousot. Systematic design of program analysis frameworks. In *Conference Record of the Sixth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 269–282, San Antonio, Texas, 1979. ACM Press, New York, New York, USA.

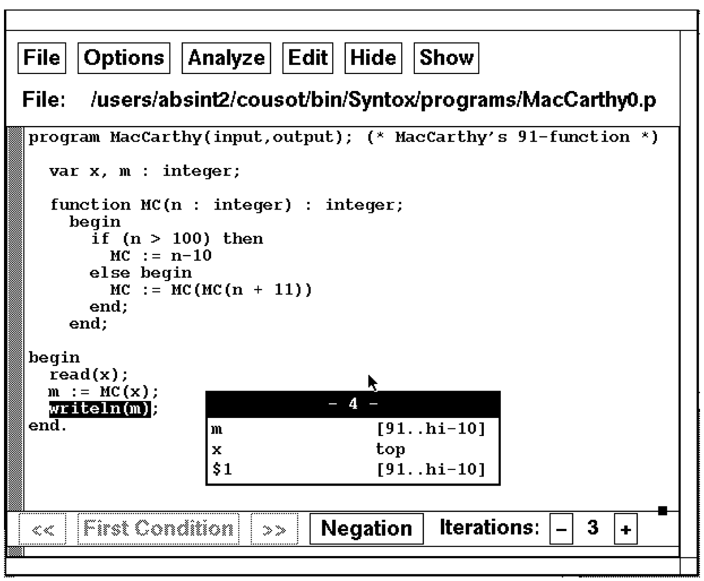
Program Analysis by Abstract Interpretation

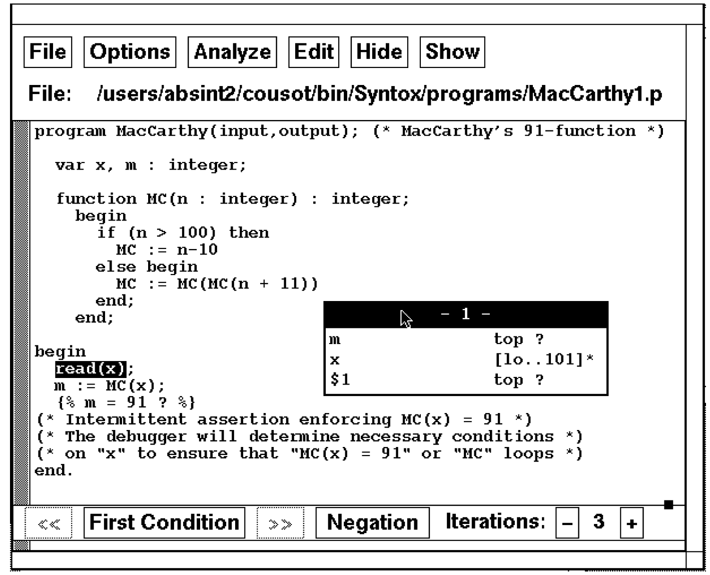
- (Bit-vector) data flow analysis;
- Strictness analysis and comportment analysis (generalizing strictness, termination, projection and PER analysis);
- Binding time analysis;
- Pointer analysis;
- Set/grammar-based analysis;
- Data dependence analysis (e.g. for vectorization/parallelization);
- Descriptive/soft and prescriptive (polymorphic) typing and type inference;
- Effect systems;
- ...

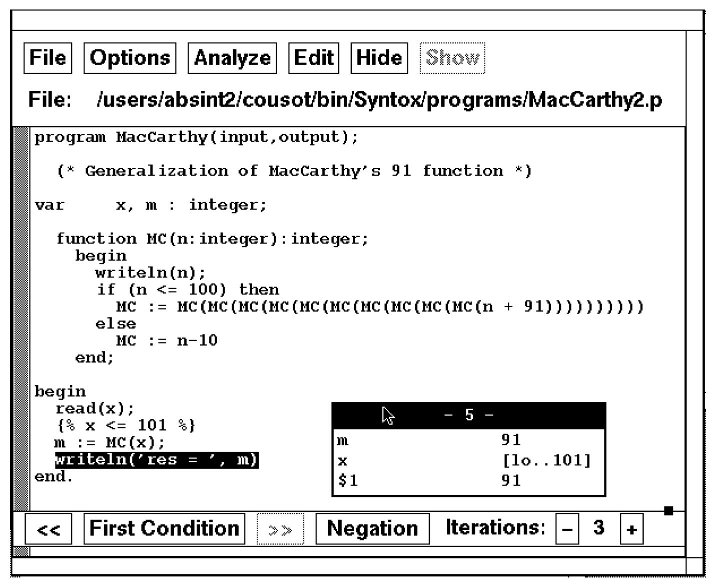
Program Debugging by Abstract Interpretation

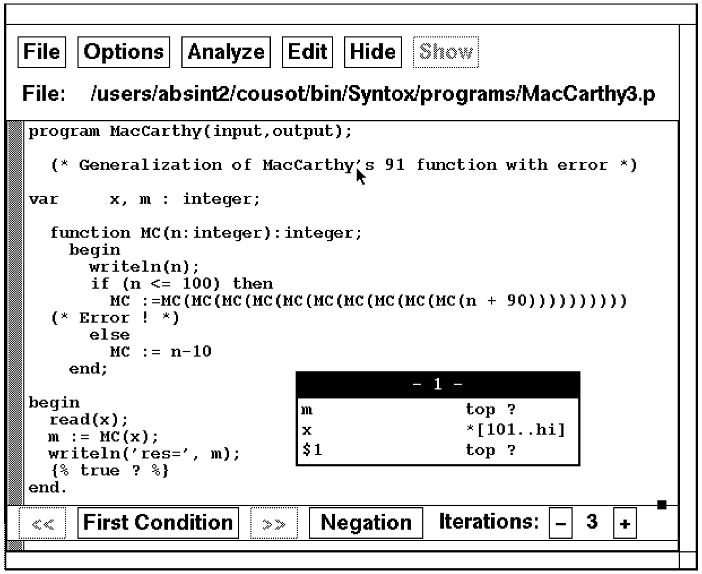
- SYNTOX [4, 5] by François Bourdoncle: interval analysis for PASCAL programs;
- For *abstract debugging*, the user can provide:
  - Invariant assertions: `{% … %}`,
  - Intermittent assertions: `{% … ? %}` (termination is required by `{% true ? %}` before final `end.`),
- At each program point the analysis provides for each numerical variable `v` a corresponding invariant interval assertion (`v [l..h]`). A star (`*`) on one of the bounds (first: `First Condition`, next: `»`, previous: `«`) indicates a *necessary* condition in the form of a run-time check to be inserted in the program for the user assertions to be satisfied. A sharp `#` indicates a possible overflow.

4 F. Bourdoncle, *Abstract Debugging of Higher-Order Imperative Languages*, Proc. PLDI'93, ACM Press, 1993, pp. 46–55.
5 http://www.ensmp.fr/ bourdonc/syntox.tar.Z



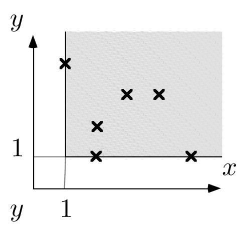




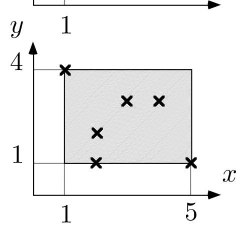


Examples of independent numerical abstractions

- Signs [6]: $x > 0 \wedge y > 0$



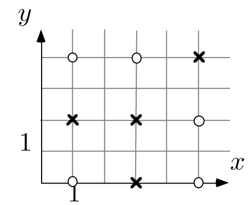
- Intervals [7]: $1 \leq x \leq 5 \wedge 1 \leq y \leq 4$



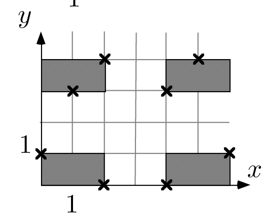
[6] P. Cousot & R. Cousot. Systematic design of program analysis frameworks. In *6th POPL*, pages 269–282, San Antonio, Texas, 1979. ACM Press.

[7] P. Cousot and R. Cousot. Static determination of dynamic properties of programs. In *Proc. 2nd International Symposium on Programming*, pages 106–130. Dunod, 1976.

- Arithmetic congruences [8]: $x = 1 \bmod 2 \wedge y = 0 \bmod 2$



- Interval congruences [9]: $x \in [0, 2] \bmod 4 \wedge y = [0, 1] \bmod 3$

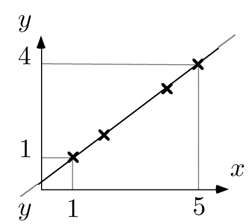


[8] P. Granger. Static analysis of arithmetical congruences. *Int. J. of Comp. Math.*, 30:165–190, 1989.

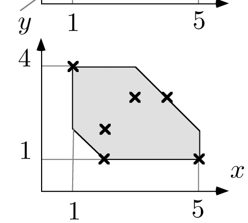
[9] F. Masdupuy. Semantic analysis of interval congruences. In D. Bjørner, M. Broy, and I.V. Pottosin, editors, *Proc. FMPA*, Academgorodok, Novosibirsk, Russia, LNCS 735, pages 142–155. Springer-Verlag, June 28–July 2, 1993.

Examples of relational numerical abstractions

- Linear equalities [10]: $-3x + 4y = 1$



- Simple sections [11]: $1 \leq x \leq 5 \wedge 1 \leq y \leq 4 \wedge 3 \leq x + y \leq 7$

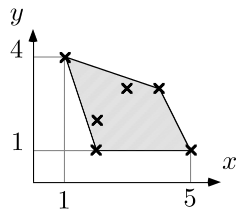


[10] M. Karr. Affine relationships among variables of a program. *Acta Inf.*, 6:133–151, 1976.

[11] V. Balasundaram and K. Kennedy. A technique for summarizing data access and its use in parallelism enhancing transformations. In *SIGPLAN'89 PLDI*, pages 41–53, Portland, Ore., June 21–23, 1989.

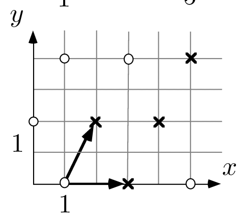
- Linear inequalities [12]:

$$\begin{array}{rrcl} & 3x + y & \geq & 7 \\ \wedge & 2x + y & \leq & 11 \\ \wedge & y & \geq & 1 \\ \wedge & x + 3y & \leq & 13 \end{array}$$



- Linear congruences [13]:

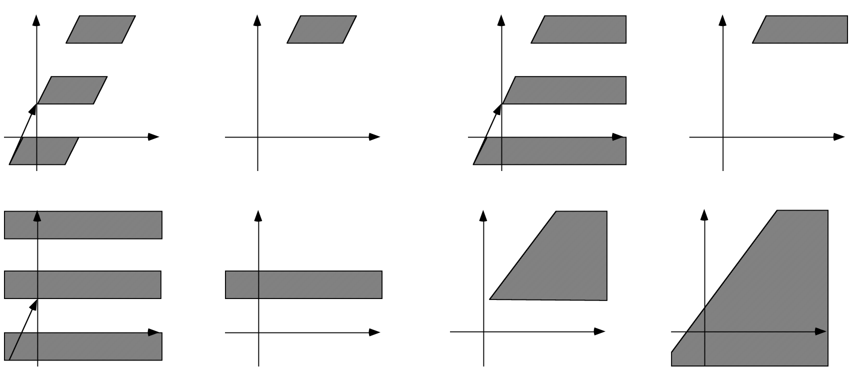
$$\begin{array}{rrcl} & 2x + y & = & 1 \bmod 2 \\ \wedge & y & = & 0 \bmod 2 \end{array}$$



[12] P. Cousot and N. Halbwachs. Automatic discovery of linear restraints among variables of a program. In 5^{th} *POPL*, pages 84–97, Tucson, Arizona, 1978. ACM Press.

[13] P. Granger. Static analysis of linear congruence equalities among variables of a program. In S. Abramsky and T.S.E. Maibaum, editors, *TAPSOFT'91, Proc. Int. Joint Conf. on Theory and Practice of Software Development*, Brighton, U.K., Volume 1 (CAAP'91), LNCS 493, pages 169–192. Springer-Verlag, 1991.

- Trapezoidal congruences [14], [15]:



[14] F. Masdupuy. Using abstract interpretation to detect array data dependencies. In *Proc. International Symposium on Supercomputing*, pages 19–27, Fukuoka, Japan, Nov. 1991. Kyushu U. Press.

[15] F. Masdupuy. Array operations abstraction using semantic analysis of trapezoid congruences. In *Proc. ACM International Conference on Supercomputing, ICS'92*, pages 226–235, Washington D.C., July 1992.

ABSTRACTION OF FIXPOINT SEMANTICS

Fixpoint Semantics Specification $\langle D, F \rangle$

- $\langle D, \sqsubseteq, \bot, \sqcup \rangle$ — Semantic domain
  - $\langle D, \sqsubseteq \rangle$ — poset
  - $\bot$ — infimum
  - $\sqcup$ — (partially defined) least upper bound
- $F \in D \overset{\mathrm{m}}{\longmapsto} D$ — Total monotone semantic transformer
- The *iterates of F from* $\bot$ are assumed to be well-defined: $F^0 \triangleq \bot$, $F^{\delta+1} = F(F^\delta)$ and $F^\lambda \triangleq \bigsqcup_{\delta<\lambda} F^\delta$, λ limit ordinal;
- The semantics is $S \triangleq \mathrm{lfp}_{\bot}^{\sqsubseteq} F = F^\epsilon$ where ϵ is the *order of the iterates* (i.e. the least ordinal such that $F(F^\epsilon) = F^\epsilon$).

Benefits of a Fixpoint Presentation of the Semantics

- Many other equivalent possible presentations [16]:
  - equational,
  - constraint,
  - closure condition,
  - rule-based,
  - game-theoretic;

[16] P. Cousot and R. Cousot. Compositional and inductive semantic definitions in fixpoint, equational, constraint, closure-condition, rule-based and game-theoretic form, invited paper. In P. Wolper, ed., *Proc. 7th Int. Conf. on Computer Aided Verification, CAV '95*, LNCS 939, pp 293–308. Springer-Verlag, 3–5 July 1995.

- Fixpoints directly lead to proof methods, e.g.:
  - Scott induction:

$$P(\bot) \;\wedge\; \forall X : P(X) \Rightarrow P(F(X)) \;\wedge\; P \text{ admissible}$$
$$\Rightarrow P(\mathrm{lfp}_{\bot}^{\sqsubseteq} F)$$

(with the hypotheses of Kleene's fixpoint theorem);

  - Park induction:

$$\mathrm{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq P$$
$$\Longleftrightarrow \;\exists I : F(I) \sqsubseteq I \;\wedge\; I \sqsubseteq P$$

(with the hypotheses of Tarski's fixpoint theorem).

- By approximation, fixpoints directly lead to iterative program analysis algorithms [17], [18];
- Fixpoint presentation of the semantics is not always possible (without further refinement of the semantic domain).

[17] P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Conference Record of the Fourth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 238–252, Los Angeles, California, 1977. ACM Press, New York, New York, USA.

[18] P. Cousot and R. Cousot. Systematic design of program analysis frameworks. In *Conference Record of the Sixth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 269–282, San Antonio, Texas, 1979. ACM Press, New York, New York, USA.

Abstraction of Fixpoint Semantics

- Concrete semantics fixpoint semantics:
  - $\langle D, \sqsubseteq \rangle$ — concrete semantic domain
  - $S[\![\tau]\!] \in D$ — concrete semantics of τ

    $\triangleq \mathrm{lfp}_{\bot}^{\sqsubseteq} F$ where $F \in D \xrightarrow{\mathrm{m}} D$ is $\sqsubseteq$-monotonic
- Abstraction function: $\alpha \in D \longmapsto D^\sharp$
- Abstract semantics fixpoint semantics:
  - $D^\sharp$ — abstract semantic domain
  - $S^\sharp[\![\tau]\!] \triangleq \alpha(S[\![\tau]\!]) \in D^\sharp$ — abstract semantics of τ
- Fixpoint characterization problem:
  - Find $\sqsubseteq^\sharp$ and $F^\sharp \in D^\sharp \xrightarrow{\mathrm{m}} D^\sharp$, $\sqsubseteq^\sharp$-monotonic such that:

$$\alpha(\mathrm{lfp}_{\bot}^{\sqsubseteq} F) = \mathrm{lfp}_{\bot^\sharp}^{\sqsubseteq^\sharp} F^\sharp$$

Kleene Fixpoint Transfer Theorem

If $\langle \mathcal{D}^\natural,\ F^\natural \rangle$ and $\langle \mathcal{D}^\sharp,\ F^\sharp \rangle$ are semantic specifications and

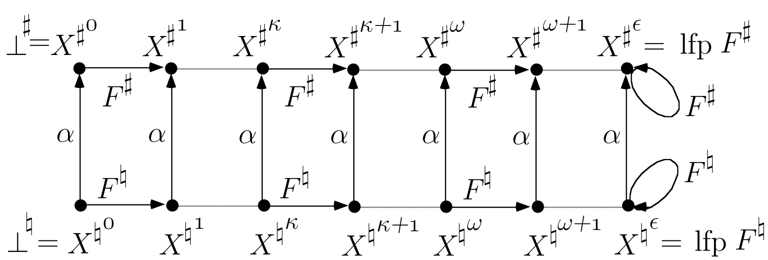
$$\alpha(\bot^\natural) \;=\; \bot^\sharp$$
$$F^\sharp \circ \alpha \;=\; \alpha \circ F^\natural$$
$$\forall\ \sqsubseteq^\natural\text{-increasing chains } X^\natural_\kappa, \kappa \in \Delta : \alpha(\bigsqcup^\natural_{\kappa\in\Delta} X^\natural_\kappa) \;=\; \bigsqcup^\sharp_{\kappa\in\Delta} \alpha(X^\natural_\kappa)$$

then

$$\alpha(\mathrm{lfp}^{\sqsubseteq^\natural} F^\natural) \;=\; \mathrm{lfp}^{\sqsubseteq^\sharp} F^\sharp$$

Note: The condition $F^\sharp \circ \alpha = \alpha \circ F^\natural$ provides guidelines for designing $F^\sharp$ when knowing $F^\natural$ and α.

Sketch of Proof of Kleene Fixpoint Transfer Theorem



Convergence

The convergence of the abstract iterates for $F^\sharp$ (at ϵ') is at least as fast as the convergence of the concrete iterates for F (at ϵ, i.e. $\epsilon' \le \epsilon$).

Proof

$$\begin{array}{rrcll}
 & F(X^{\natural^\epsilon}) & = & X^{\natural^\epsilon} & \text{hypothesis} \\
\Rightarrow & \alpha(F(X^{\natural^\epsilon})) & = & \alpha(X^{\natural^\epsilon}) & \\
\Rightarrow & F^\sharp(\alpha(X^{\natural^\epsilon})) & = & \alpha(X^{\natural^\epsilon}) & \text{since } F^\sharp \circ \alpha = \alpha \circ F^\natural \\
\Rightarrow & F^\sharp(X^{\sharp^\epsilon}) & = & X^{\sharp^\epsilon} & \text{since } X^{\sharp^\epsilon} = \alpha(X^{\natural^\epsilon}) \\
\Rightarrow & \epsilon' \le \epsilon & & &
\end{array}$$

□

Abstraction function

- An important particular case of abstraction function:

$$\alpha \;\in\; \langle \mathcal{D}^\natural,\ \sqsubseteq^\natural \rangle \longmapsto \langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp \rangle$$

  is when α preserves existing lubs:

$$\alpha\Big(\bigsqcup^\natural_{i\in\Delta} x_i\Big) \;=\; \bigsqcup^\sharp_{i\in\Delta} \alpha(x_i)$$

- In this case there exists a unique $\gamma \in \langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp \rangle \longmapsto \langle \mathcal{D}^\natural,\ \sqsubseteq^\natural \rangle$ such that the pair $\langle \alpha,\ \gamma \rangle$ is a Galois connection.

Galois Connection

Given posets $\langle \mathcal{D}^\natural,\ \sqsubseteq^\natural \rangle$ and $\langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp \rangle$, a *Galois connection* is a pair of maps such that:

$$\alpha \in \mathcal{D}^\natural \longmapsto \mathcal{D}^\sharp$$

$$\gamma \in \mathcal{D}^\sharp \longmapsto \mathcal{D}^\natural$$

$$\forall x \in \mathcal{D}^\natural : \forall y \in \mathcal{D}^\sharp : \alpha(x) \sqsubseteq^\sharp y \Leftrightarrow x \sqsubseteq^\natural \gamma(y)$$

in which case we write:

$$\langle \mathcal{D}^\natural,\ \sqsubseteq^\natural \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp \rangle$$

If α is surjective then we have a *Galois insertion* and write:

$$\langle \mathcal{D}^\natural,\ \sqsubseteq^\natural \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows\!\!\!\!\rightarrow}} \langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp \rangle$$

Example of Galois Connection: Elementwise Abstraction

- If
  - $\mathtt{@} \in \mathcal{D}^\natural \longmapsto \mathcal{D}^\sharp$
  - $\alpha \in \wp(\mathcal{D}^\natural) \longmapsto \wp(\mathcal{D}^\sharp)$
    $\alpha(X) \stackrel{\Delta}{=} \{\mathtt{@}(x) \mid x \in X\}$
  - $\gamma \in \wp(\mathcal{D}^\sharp) \longmapsto \wp(\mathcal{D}^\natural)$
    $\gamma(Y) \stackrel{\Delta}{=} \{x \mid \mathtt{@}(x) \in Y\}$

  then

$$\langle \wp(\mathcal{D}^\natural),\ \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \wp(\mathcal{D}^\sharp),\ \subseteq \rangle$$

If $\mathtt{@}$ is surjective then so is α.

Proof $\alpha(X) \subseteq Y \Leftrightarrow \{\mathtt{@}(x) \mid x \in X\} \subseteq Y \Leftrightarrow \forall x \in X : \mathtt{@}(x) \in Y \Leftrightarrow X \subseteq \{x \mid \mathtt{@}(x) \in Y\} \Leftrightarrow X \subseteq \gamma(Y)$. □

Tarski Fixpoint Transfer Theorem

If $\langle \mathcal{D}^\natural,\ \sqsubseteq^\natural,\ \bot^\natural,\ \sqcup^\natural \rangle$ and $\langle \mathcal{D}^\sharp,\ \sqsubseteq^\sharp,\ \bot^\sharp,\ \sqcup^\sharp \rangle$ are complete lattices, $F^\natural \in \mathcal{D}^\natural \overset{\mathrm{m}}{\longmapsto} \mathcal{D}^\natural$, $F^\sharp \in \mathcal{D}^\sharp \overset{\mathrm{m}}{\longmapsto} \mathcal{D}^\sharp$ are monotonic and

- α is a complete $\sqcap$-morphism (a)
- $F^\sharp \circ \alpha \sqsubseteq^\sharp \alpha \circ F^\natural$ (b)
- $\forall y \in \mathcal{D}^\sharp : F^\sharp(y) \sqsubseteq^\sharp y \Rightarrow \exists x \in \mathcal{D}^\natural : \alpha(x) = y \wedge F^\natural(x) \sqsubseteq^\natural x$ (c)

then

$$\alpha(\mathrm{lfp}^{\sqsubseteq^\natural} F^\natural) = \mathrm{lfp}^{\sqsubseteq^\sharp} F^\sharp$$

Proof

(d) $F^\natural(x) \sqsubseteq^\natural x$
$\Rightarrow \alpha \circ F^\natural(x) \sqsubseteq^\natural \alpha(x)$ since α is monotonic by (a)
$\Rightarrow F^\sharp \circ \alpha(x) \sqsubseteq^\natural \alpha(x)$ by (b)

(e) $\{\alpha(x) \mid F^\natural(x) \sqsubseteq^\natural x\} = \{y \mid F^\sharp(y) \sqsubseteq^\sharp y\}$ by (c) and (d)

(f) $\sqcap^\sharp\{\alpha(x) \mid F^\natural(x) \sqsubseteq^\natural x\} = \sqcap^\sharp\{y \mid F^\sharp(y) \sqsubseteq^\sharp y\}$ by (e)
$\Rightarrow \alpha(\sqcap^\natural\{x \mid F^\natural(x) \sqsubseteq^\natural x\}) = \sqcap^\sharp\{y \mid F^\sharp(y) \sqsubseteq^\sharp y\}$ by (a)
$\Rightarrow \alpha(\mathrm{lfp}^{\sqsubseteq^\natural} F^\natural) = \mathrm{lfp}^{\sqsubseteq^\sharp} F^\sharp$ by Tarski's fixpt th.

□

Trace semantics

Transition System

- A transition system is a pair $\langle \Sigma,\ \tau \rangle$ where:
  - Σ is a (non-empty) set of states,
  - We could also consider actions as in process algebra,
  - $\tau \subseteq \Sigma \times \Sigma$ is the binary transition relation between a state and its possible successors;
- We write $s\ \tau\ s'$ or $\tau(s, s')$ for $\langle s,\ s' \rangle \in \tau$ using the isomorphism $\wp(\Sigma \times \Sigma) \simeq (\Sigma \times \Sigma) \longmapsto \mathbb{B}$;
- $\mathbb{B} \stackrel{\Delta}{=} \{\mathrm{tt}, \mathrm{ff}\}$ is the set of boolean values;
- $\check{\tau} \stackrel{\Delta}{=} \{s \in \Sigma \mid \forall s' \in \Sigma : \neg(s\ \tau\ s')\}$ is the set of final/blocking states.

Sequences

Finite Sequences

- $\mathcal{A}$ — non-empty alphabet
- $\mathcal{A}^{\vec{0}} \stackrel{\Delta}{=} \{\vec{\varepsilon}\}$ — empty sequence
- $\mathcal{A}^{\vec{n}} \stackrel{\Delta}{=} [0,\ n-1] \longmapsto \mathcal{A}$ when $n > 0$ — finite sequences of length n
- $\mathbb{N}_+ \stackrel{\Delta}{=} \{n \in \mathbb{N} \mid n > 0\}$ — positive naturals
- $\mathcal{A}^{\vec{+}} \stackrel{\Delta}{=} \bigcup_{n \in \mathbb{N}_+} \mathcal{A}^{\vec{n}}$ — non-empty finite sequences
- $\mathcal{A}^{\vec{\star}} \stackrel{\Delta}{=} \mathcal{A}^{\vec{+}} \cup \{\vec{\varepsilon}\}$ — finite sequences
- The length of a finite sequence $\sigma \in \mathcal{A}^{\vec{n}}$ is $|\sigma| \stackrel{\Delta}{=} n$;

Infinite Sequences

- $\mathcal{A}^{\vec{\omega}} \stackrel{\Delta}{=} \mathbb{N} \longmapsto \mathcal{A}$ — infinite sequences
- $\mathcal{A}^{\vec{\infty}} \stackrel{\Delta}{=} \mathcal{A}^{\vec{\star}} \cup \mathcal{A}^{\vec{\omega}}$ — sequences
- $\mathcal{A}^{\vec{\propto}} \stackrel{\Delta}{=} \mathcal{A}^{\vec{+}} \cup \mathcal{A}^{\vec{\omega}}$ — non-empty sequences
- The length of an infinite sequence $\sigma \in \mathcal{A}^{\vec{\omega}}$ is $|\sigma| \stackrel{\Delta}{=} \omega$

Junction of Finite Sequences

- Joinable non-empty finite sequences:

$$\alpha_0 \ldots \alpha_{\ell-1} \mathrel{?} \beta_0 \ldots \beta_{m-1} \text{ iff } \alpha_{\ell-1} = \beta_0$$

- Their join is:

$$\frac{\begin{array}{cccccc} & & \alpha_0 \;\ldots\; \alpha_{\ell-1} & & \\ & & = & & \\ & & \beta_0 & \beta_1 \;\ldots\; \beta_{m-1} \end{array}}{\alpha_0 \ldots \alpha_{\ell-1} \frown \beta_0 \ldots \beta_{m-1} \stackrel{\Delta}{=} \alpha_0 \;\ldots\; \alpha_{\ell-1}\; \beta_1 \;\ldots\; \beta_{m-1}}$$

Junction of Infinitary Sequences

- Joinable infinitary sequences:

$$\alpha_0 \ldots \alpha_\ell \ldots \mathrel{?} \beta_0 \ldots \beta_{m-1} \text{ is true}$$
$$\alpha_0 \ldots \alpha_\ell \ldots \mathrel{?} \beta_0 \ldots \beta_m \ldots \text{ is true}$$
$$\alpha_0 \ldots \alpha_{\ell-1} \mathrel{?} \beta_0 \ldots \beta_m \ldots \text{ iff } \alpha_{\ell-1} = \beta_0$$

- Their join is:

$$\alpha_0 \ldots \alpha_\ell \ldots \frown \beta_0 \ldots \beta_{m-1} \stackrel{\Delta}{=} \alpha_0 \;\ldots\; \alpha_\ell \;\ldots$$
$$\alpha_0 \ldots \alpha_\ell \ldots \frown \beta_0 \ldots \beta_m \ldots \stackrel{\Delta}{=} \alpha_0 \;\ldots\; \alpha_\ell \;\ldots$$
$$\frac{\begin{array}{cc} \alpha_0 \;\ldots\; \alpha_{\ell-1} & \\ = & \\ \beta_0 & \beta_1 \;\ldots\; \beta_m \ldots \end{array}}{\alpha_0 \ldots \alpha_{\ell-1} \frown \beta_0 \ldots \beta_m \ldots \stackrel{\Delta}{=} \alpha_0 \;\ldots\; \alpha_{\ell-1}\; \beta_1 \;\ldots\; \beta_m \ldots}$$

Junction of Sets of Sequences

- For sets A and $B \in \wp(\mathcal{A}^{\vec{\propto}})$ of non-empty sequences, we have:
  - $A \frown B \stackrel{\Delta}{=} \{\alpha \frown \beta \mid \alpha \in A \wedge \beta \in B \wedge \alpha \mathrel{?} \beta\}$ junction
- $A \frown \Big(\bigcup_{i \in \Delta} B_i\Big) = \bigcup_{i \in \Delta} (A \frown B_i)$ and $\Big(\bigcup_{i \in \Delta} A_i\Big) \frown B = \bigcup_{i \in \Delta} (A_i \frown B)$
- Not co-continuous on $\wp(\mathcal{A}^{\vec{\propto}})$! Counter example ($\mathcal{A} = \{a\}$):
  - $A = \{a^\omega\}$,
  - $B_n = \{a^\ell \mid \ell \in \mathbb{N} \wedge \ell > n\}$, $n \in \mathbb{N}$ is a $\subseteq$-decreasing chain,
  - $A \frown \Big(\bigcap_{n \in \mathbb{N}} B_n\Big) = \emptyset$ and $\Big(\bigcap_{n \in \mathbb{N}} A \frown B_n\Big) = \{a^\omega\}$.

Trace Semantics

- $\langle \Sigma,\ \tau \rangle$ — transition system
- $\tau^{\dot{\vec{n}}} \triangleq \{\sigma \in \Sigma^{\vec{n}} \mid \forall i < n-1 : \sigma_i \ \tau \ \sigma_{i+1}\}$ — partial traces of length n
- $\check{\tau} \triangleq \{s \in \Sigma \mid \forall s' \in \Sigma : \neg(s \ \tau \ s')\}$ — final/blocking states
- $\tau^{\vec{n}} \triangleq \{\sigma \in \tau^{\dot{\vec{n}}} \mid \sigma_{n-1} \in \check{\tau}\}$ — complete traces of length n
- $\tau^{\vec{+}} \triangleq \bigcup_{n \in \mathbb{N}_+} \tau^{\vec{n}}$ — finite complete traces
- $\tau^{\vec{\omega}} \triangleq \{\sigma \in \Sigma^{\vec{\omega}} \mid \forall i \in \mathbb{N} : \sigma_i \ \tau \ \sigma_{i+1}\}$ — infinite traces
- $\tau^{\vec{\infty}} \triangleq \tau^{\vec{+}} \cup \tau^{\vec{\omega}}$ — complete traces

Fixpoint Characterization of $\tau^{\vec{+}}$ (finite complete execution traces)

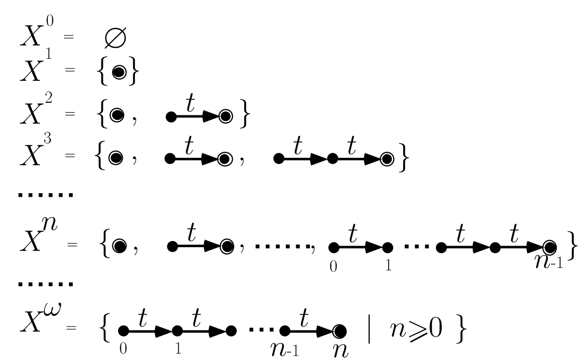
$$\tau^{\vec{+}} = \mathrm{lfp}_{\emptyset}^{\subseteq} F^{\vec{+}}$$

where the set of finite traces transformer $F^{\vec{+}}$ is:

$$F^{\vec{+}}(X) \triangleq \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X$$

Note: $F^{\vec{+}}$ is a complete $\cup$-morphism: $\bigcup_i F^{\vec{+}}(X_i) = F^{\vec{+}}(\bigcup_i X_i)$.

Sketch of Proof of $\mathrm{lfp}_{\emptyset}^{\subseteq} F^{\vec{+}} = \bigcup_{i \in \mathbb{N}_+} \tau^{\vec{i}} = \tau^{\vec{+}}$



Fixpoint Characterization of $\tau^{\vec{\omega}}$ (infinite execution traces)

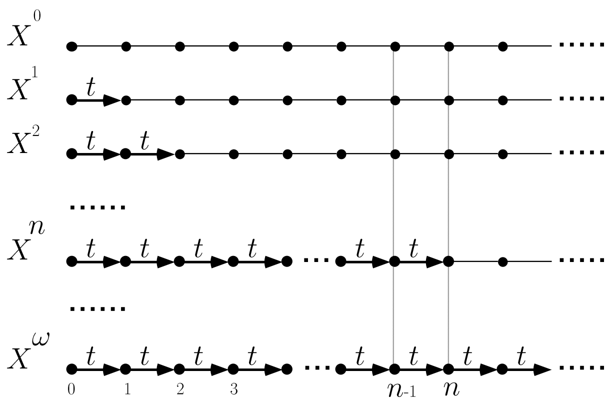
$$\tau^{\vec{\omega}} = \mathrm{gfp}_{\Sigma^{\vec{\omega}}}^{\subseteq} F^{\vec{\omega}}$$

where the set of infinite traces transformer $F^{\vec{\omega}}$ is:

$$F^{\vec{\omega}}(X) \triangleq \tau^{\dot{\vec{2}}} \frown X$$

Note: $F^{\vec{\omega}}$ is a complete $\cap$-morphism: $\bigcap_i F^{\vec{\omega}}(X_i) = F^{\vec{\omega}}(\bigcap_i X_i)$.

Sketch of Proof of $\text{gfp}^{\subseteq}_{\Sigma^{\vec{\omega}}} F^{\vec{\omega}} = \bigcap_{n\in\mathbb{N}} \tau^{\dot{\vec{n}}} \frown \Sigma^{\vec{\omega}} = \tau^{\vec{\omega}}$



(Trivial) bi-fixpoint theorem

If
- $\Sigma^{\vec{+}}$, $\Sigma^{\vec{\omega}}$ is a partition of $\Sigma^{\vec{\infty}}$
- $\langle \wp(\Sigma^{\vec{+}}),\ \sqsubseteq^{\vec{+}},\ \bot^{\vec{+}},\ \sqcup^{\vec{+}}\rangle$ (resp. $\langle \wp(\Sigma^{\vec{\omega}}),\ \sqsubseteq^{\vec{\omega}},\ \bot^{\vec{\omega}},\ \sqcup^{\vec{\omega}}\rangle$) is a complete lattice (resp. cpo)
- $F^{\vec{+}} \in \wp(\Sigma^{\vec{+}}) \overset{\text{m}}{\longmapsto} \wp(\Sigma^{\vec{+}})$ (resp. $F^{\vec{\omega}} \in \wp(\Sigma^{\vec{\omega}}) \overset{\text{m}}{\longmapsto} \wp(\Sigma^{\vec{\omega}})$) is monotonic (resp. continuous, a complete join morphism)
- $X^{\vec{+}} \triangleq X \cap \Sigma^{\vec{+}}$, $X^{\vec{\omega}} \triangleq X \cap \Sigma^{\vec{\omega}}$
- $F^{\vec{\infty}}(X) \triangleq F^{\vec{+}}(X^{\vec{+}}) \cup F^{\vec{\omega}}(X^{\vec{\omega}})$
- $X \sqsubseteq^{\vec{\infty}} Y \triangleq X^{\vec{+}} \sqsubseteq^{\vec{+}} Y^{\vec{+}} \wedge X^{\vec{\omega}} \sqsubseteq^{\vec{\omega}} Y^{\vec{\omega}}$
- $\bot^{\vec{\infty}} \triangleq \bot^{\vec{+}} \cup \bot^{\vec{\omega}}$
- $\bigsqcup^{\vec{\infty}}_i X_i \triangleq \bigsqcup^{\vec{+}}_i X_i^{\vec{+}} \cup \bigsqcup^{\vec{\omega}}_i X_i^{\vec{\omega}}$

then:
- $\langle \wp(\Sigma^{\vec{\infty}}),\ \sqsubseteq^{\vec{\infty}},\ \bot^{\vec{\infty}},\ \sqcup^{\vec{\infty}}\rangle$ is a complete lattice (resp. cpo)
- $F^{\vec{\infty}}$ is monotonic (resp. continuous, a complete join morphism)
- $\text{lfp}^{\sqsubseteq^{\vec{\infty}}}_{\bot^{\vec{\infty}}} F^{\vec{\infty}} = \text{lfp}^{\sqsubseteq^{\vec{+}}}_{\bot^{\vec{\infty}}} F^{\vec{+}} \cup \text{lfp}^{\sqsubseteq^{\vec{\omega}}}_{\bot^{\vec{\infty}}} F^{\vec{\omega}}$

Approximation and Computational Orderings

- $\langle \wp(\Sigma^{\vec{\infty}}),\ \sqsubseteq^{\vec{\infty}},\ \bot^{\vec{\infty}},\ \sqcup^{\vec{\infty}}\rangle$ is a complete lattice (or cpo) for the *computational ordering* $\sqsubseteq^{\vec{\infty}}$;
- $\langle \wp(\Sigma^{\vec{\infty}}),\ \subseteq,\ \emptyset,\ \cup\rangle$ is a complete lattice for the *approximation ordering* $\subseteq$ (logical implication);
- Sometimes further abstractions identify $\sqsubseteq^{\vec{\infty}}$ and $\subseteq$ (e.g. strictness analysis).

Fixpoint Characterization of $\tau^{\vec{\infty}}$ (complete execution traces)

$$\tau^{\vec{\infty}} = \tau^{\vec{+}} \cup \tau^{\vec{\omega}} = \mathrm{lfp}_{\emptyset}^{\subseteq} F^{\vec{+}} \;\cup\; \mathrm{lfp}_{\Sigma^{\vec{\omega}}}^{\supseteq} F^{\vec{\omega}} = \mathrm{lfp}_{\Sigma^{\vec{\infty}}}^{\sqsubseteq^{\vec{\infty}}} F^{\vec{\infty}}$$

by the bifixpoint theorem where the set of complete traces transformer $F^{\vec{\infty}}$ is:

$$F^{\vec{\infty}}(X) \stackrel{\Delta}{=} \tau^{\vec{1}} \;\cup\; \tau^{\dot{\vec{2}}} \frown X$$

Proof

$$\begin{aligned}
F^{\vec{\infty}}(X) &\stackrel{\Delta}{=} F^{\vec{+}}(X^{\vec{+}}) \cup F^{\vec{\omega}}(X^{\vec{\omega}}) \\
&= \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X^{\vec{+}} \cup \tau^{\dot{\vec{2}}} \frown X^{\vec{\omega}} \\
&= \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown (X^{\vec{+}} \cup X^{\vec{\omega}}) \\
&= \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X
\end{aligned}$$

□

Continuity of the trace transformer $F^{\vec{\infty}}(X)$

Unbounded non-determinism does not imply absence of continuity of the transformer of the fixpoint semantics:

Proof

$$\begin{aligned}
\bigsqcup_i^{\vec{\infty}} F^{\vec{\infty}}(X_i) &= \bigsqcup_i^{\vec{\infty}} \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X_i \\
&= \bigcup_i (\tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X_i^{\vec{+}}) \cup \bigcap_i (\tau^{\dot{\vec{2}}} \frown X_i^{\vec{\omega}}) \\
&= \tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown (\bigcup_i X_i^{\vec{+}} \cup \bigcap_i X_i^{\vec{\omega}}) \\
&= F^{\vec{\infty}}(\bigsqcup_i^{\vec{\infty}} X_i)
\end{aligned}$$

□

Scott's thesis (slightly revisited)

The semantics of a program can be expressed as the least fixpoint of a continuous operator (even in presence of unbounded nondeterminism), *for a sufficiently refined semantic domain.*

Transition Versus Trace Semantics

Maximal Trace Semantics/Transition Semantics

The transition/small-step operational semantics is an abstraction of the maximal trace semantics:

$$\tau = \alpha^{\tau}(\tau^{\vec{\infty}})$$

where

- the abstraction collects possible transitions $\alpha^{\tau}(T) \stackrel{\Delta}{=} \{\langle s,\ s'\rangle \mid \exists \sigma \in \Sigma^{\vec{\star}} : \exists \sigma' \in \Sigma^{\vec{\propto}} : \sigma \cdot ss' \cdot \sigma' \in T\}$;
- the concretization builds maximal execution traces $\gamma^{\tau}(t) \stackrel{\Delta}{=} t^{\vec{\infty}}$;
- $\langle \wp(\Sigma^{\vec{\infty}}),\ \subseteq\rangle \underset{\alpha^{\tau}}{\overset{\gamma^{\tau}}{\longleftarrow\!\!\!\longrightarrow\!\!\!\!\!\rightarrow}} \langle \wp(\Sigma \times \Sigma),\ \subseteq\rangle$.

The Transition Abstraction is Approximate

In general:

$$T \subsetneq \gamma^{\tau}(\alpha^{\tau}(T))$$

Counter-example:

- set of fair traces $T = \{a^n b \mid n \in \mathbb{N}\}$
- $\alpha^{\tau}(T) = \{\langle a,\ a\rangle, \langle a,\ b\rangle\}$
- $\gamma^{\tau}(\alpha^{\tau}(T)) = \{a^n b \mid n \in \mathbb{N}\} \cup \{a^{\omega}\}$ is unfair for b.

Relational Semantics

Finite Relational Abstraction

Replace finite execution traces $\sigma_0\sigma_1 \ldots \sigma_{n-1}$ by their initial/final states $\langle \sigma_0,\ \sigma_{n-1}\rangle$:

- $@^{+} \in \Sigma^{\vec{+}} \longmapsto (\Sigma \times \Sigma)$
  $@^{+}(\sigma) \stackrel{\Delta}{=} \langle \sigma_0,\ \sigma_{n-1}\rangle, \quad n \in \mathbb{N}_+,\ \sigma \in \Sigma^{\vec{n}}$
- $\alpha^{+}(X) \stackrel{\Delta}{=} \{@^{+}(\sigma) \mid \sigma \in X\}$
  $\gamma^{+}(Y) \stackrel{\Delta}{=} \{\sigma \mid @^{+}(\sigma) \in Y\}$
- $\langle \wp(\Sigma^{\vec{+}}),\ \subseteq\rangle \underset{\alpha^{+}}{\overset{\gamma^{+}}{\longleftarrow\!\!\!\longrightarrow\!\!\!\!\!\rightarrow}} \langle \wp(\Sigma \times \Sigma),\ \subseteq\rangle$

Finitary Relational Semantics of a Transition System $\langle \Sigma,\ \tau \rangle$

- Finitary relational / big-step operational / natural semantics:

$$\tau^{+} \stackrel{\Delta}{=} \alpha^{+}(\tau^{\vec{+}}) = \alpha^{+}(\mathrm{lfp}_{\emptyset}^{\subseteq} F^{\vec{+}})$$

- Fixpoint characterization:

$$\begin{aligned} \tau^{+} &= \mathrm{lfp}_{\emptyset}^{\subseteq} F^{+} \\ F^{+}(X) &\stackrel{\Delta}{=} \check{\tau} \cup \tau \circ X \\ \check{\tau} &\stackrel{\Delta}{=} \{\langle s,\ s \rangle \in \Sigma \mid \forall s' \in \Sigma : \neg(s\ \tau\ s')\} \end{aligned}$$

Proof

- $\alpha^{+}(\emptyset) \stackrel{\Delta}{=} \{@^{+}(\sigma) \mid \sigma \in \emptyset\} = \emptyset$
- $\alpha^{+} \circ F^{\vec{+}} = \lambda X \cdot \alpha^{+}(\tau^{\vec{1}} \cup \tau^{\dot{\vec{2}}} \frown X)$

$$\begin{aligned} &= \lambda X \cdot \alpha^{+}(\tau^{\vec{1}}) \cup \alpha^{+}(\tau^{\dot{\vec{2}}} \frown X) \\ &= \lambda X \cdot \{\langle s,\ s \rangle \in \Sigma \mid \forall s' \in \Sigma : \neg(s\ \tau\ s')\} \cup \alpha^{+}(\tau^{\dot{\vec{2}}} \frown X) \\ &= \lambda X \cdot \check{\tau} \cup \{@^{+}(\eta \frown \xi) \mid \eta \in \tau^{\dot{\vec{2}}} \wedge \xi \in X \wedge \eta \mathrel{?} \xi\} \\ &= \lambda X \cdot \check{\tau} \cup \{\langle \eta_0,\ \xi_{n-1} \rangle \mid \eta_0\ \tau\ \xi_0 \wedge n \in \mathbb{N}_{+} \wedge \xi \in X \cap \Sigma^{\vec{n}}\} \\ &= \lambda X \cdot \check{\tau} \cup \{\langle s,\ s' \rangle \mid \exists s'' : s\ \tau\ s'' \wedge \langle s'',\ s' \rangle \in \alpha^{+}(X)\} \\ &= \lambda X \cdot \check{\tau} \cup \tau \circ \alpha^{+}(X) \\ &= F^{+} \circ \alpha^{+} \end{aligned}$$

- α^{+} is continuous (Galois connection)
- $\tau^{+} = \alpha^{+}(\mathrm{lfp}_{\emptyset}^{\subseteq} F^{\vec{+}}) = \mathrm{lfp}_{\emptyset}^{\subseteq} F^{+}$ by Kleene's fixpoint transfer th.

□

- α^{+} is a $\cap$-morphism but not co-continuous hence not a complete $\cap$-morphism.

Proof

- $X^{k} \stackrel{\Delta}{=} \{a^{n}b \mid n \geq k\}$
- $X^{k}, k \in \mathbb{N}_{+}$ is $\subseteq$-decreasing
- $\bigcap_{k \in \mathbb{N}_{+}} \alpha^{+}(X^{k}) = \bigcap_{k \in \mathbb{N}_{+}} \{\langle a,\ b \rangle\} = \{\langle a,\ b \rangle\}$
- $\bigcap_{k \in \mathbb{N}_{+}} X^{k} = \emptyset$ since $a^{n}b \in \bigcap_{k \in \mathbb{N}_{+}} X^{k}$ for $n \in \mathbb{N}_{+}$ is in contradiction with $a^{n}b \notin X^{n+1}$
- $\alpha^{+}(\bigcap_{k \in \mathbb{N}_{+}} X^{k}) = \alpha^{+}(\emptyset) = \emptyset$

□

- It follows that Tarski fixpoint transfer would not have been applicable.

Infinitary Relational Abstraction

Replace infinite execution traces $\sigma_0\sigma_1 \ldots \sigma_n \ldots$ by their initial state $\langle \sigma_0,\ \bot \rangle$, making non-termination by Scott's $\bot$:

- $@^{\omega} \in \Sigma^{\vec{\omega}} \longmapsto \Sigma \times \{\bot\}$ [19]

  $\bot \notin \Sigma$ non-termination notation

  $@^{\omega}(\sigma) \stackrel{\Delta}{=} \langle \sigma_0,\ \bot \rangle,\ \sigma \in \Sigma^{\vec{\omega}}$
- $\alpha^{\omega}(X) \stackrel{\Delta}{=} \{@^{\omega}(\sigma) \mid \sigma \in X\}$

  $\gamma^{\omega}(Y) \stackrel{\Delta}{=} \{\sigma \mid @^{\omega}(\sigma) \in Y\}$
- $\langle \wp(\Sigma^{\vec{\omega}}),\ \subseteq \rangle \underset{\alpha^{\omega}}{\overset{\gamma^{\omega}}{\leftrightarrows}} \langle \wp(\Sigma \times \{\bot\}),\ \subseteq \rangle$

[19] or isomorphically $\alpha^{\omega} \in \wp(\Sigma^{\vec{\omega}}) \longmapsto \wp(\Sigma)$.

- α^ω is a complete $\cup$-morphism (Galois connection, hence continuous) and a $\cap$-morphism but not co-continuous.

Proof

- $X^k \triangleq \{a^n b^\omega \mid n \geq k\}$
- $X^k, k \in \mathbb{N}_+$ is $\subseteq$-decreasing
- $\bigcap_{k\in\mathbb{N}_+} \alpha^\omega(X^k) = \bigcap_{k\in\mathbb{N}_+} \{\langle a,\ \bot\rangle\} = \{\langle a,\ \bot\rangle\}$
- $\bigcap_{k\in\mathbb{N}_+} X^k = \emptyset$ since $a^n b^\omega \in \bigcap_{k\in\mathbb{N}_+} X^k$ for $n \in \mathbb{N}_+$ is in contradiction with $a^n b^\omega \notin X^{n+1}$
- $\alpha^\omega(\bigcap_{k\in\mathbb{N}_+} X^k) = \alpha^\omega(\emptyset) = \emptyset$

□

- It follows that Kleene dual fixpoint transfer does not apply.

Infinitary Relational Semantics of a Transition System $\langle\Sigma,\ \tau\rangle$

- Infinitary relational semantics:

$$\tau^\omega \triangleq \alpha^\omega(\tau^{\vec{\omega}}) = \alpha^\omega(\mathrm{gfp}^{\subseteq}_{\Sigma^{\vec{\omega}}} F^{\vec{\omega}}) = \alpha^\omega(\mathrm{lfp}^{\supseteq}_{\Sigma^{\vec{\omega}}} F^{\vec{\omega}})$$

- Fixpoint characterization:

$$\tau^\omega = \mathrm{lfp}^{\supseteq}_{\Sigma\times\{\bot\}} F^\omega = \mathrm{gfp}^{\subseteq}_{\Sigma\times\{\bot\}} F^\omega$$
$$F^\omega(X) = \tau \circ X$$

Proof

- α^ω is a complete $\cup$-morphism (G.c.) hence a complete meet morphism for $\supseteq$.
- $$\begin{aligned}\alpha^\omega \circ F^{\vec{\omega}} &= \lambda X\boldsymbol{\cdot}\alpha^\omega(\tau^{\dot{\vec{2}}} \frown X)\\ &= \lambda X\boldsymbol{\cdot}\{@^\omega(\eta \frown \xi) \mid \eta \in \tau^{\dot{\vec{2}}} \wedge \xi \in X \wedge \eta \mathbin{?} \xi\}\\ &= \lambda X\boldsymbol{\cdot}\{\langle \eta_0,\ \bot\rangle \mid \eta_0\ \tau\ \xi_0 \wedge \xi \in X\}\\ &= \lambda X\boldsymbol{\cdot}\{\langle s,\ \bot\rangle \mid \exists s' : s\ \tau\ s' \wedge \langle s',\ \bot\rangle \in \alpha^\omega(X)\}\\ &= \lambda X\boldsymbol{\cdot}\tau \circ \alpha^\omega(X)\\ &= F^\omega \circ \alpha^\omega\end{aligned}$$
- We prove that $\forall Y \in \wp(\Sigma \times \{\bot\}) : F^\omega(Y) \supseteq Y \Rightarrow \exists X \in \Sigma^{\vec{\omega}} : \alpha^\omega(X) = Y \wedge F^{\vec{\omega}}(X) \supseteq X$:
  - $X \triangleq \{\sigma \in \tau^{\vec{\omega}} \mid \forall i \in \mathbb{N} : \langle \sigma_i,\ \bot\rangle \in Y\}$
  - We first prove that $\alpha^\omega(X) = Y$:
    * $\alpha^\omega(X) \subseteq Y$ is obvious since $\sigma \in X$ implies $\langle \sigma_0,\ \bot\rangle \in Y$.

    * $Y \subseteq \alpha^\omega(X)$

    (a) $Y \subseteq F^\omega(Y) = \tau \circ Y = \{\langle s,\ \bot\rangle \mid \exists s' : s\ \tau\ s' \wedge \langle s',\ \bot\rangle \in Y\}$

    (b) If $\sigma_0 \ldots \sigma_n$ is such that $\sigma_i\ \tau\ \sigma_{i+1}$, $i < n$ and $\langle \sigma_i,\ \bot\rangle \in Y$, $i \leq n$ then $\langle \sigma_n,\ \bot\rangle \in Y$ and (a) imply $\exists \sigma_{n+1}$: $\sigma_n\ \tau\ \sigma_{n+1} \wedge \langle \sigma_{n+1},\ \bot\rangle \in Y$. So, by induction, we can built $\sigma \in \tau^{\vec{\omega}}$ such that $\forall i \in \mathbb{N}$: $\langle \sigma_i,\ \bot\rangle \in Y$. We have $\sigma \in X$ and $\langle \sigma_0,\ \bot\rangle \in \alpha^\omega(X)$ proving that $Y \subseteq \alpha^\omega(X)$;
  - Next, we prove $F^{\vec{\omega}}(X) \supseteq X$: $F^{\vec{\omega}}(X) \supseteq X \iff X \subseteq \tau^{\dot{\vec{2}}} \frown X \iff \forall \sigma \in X{:}\sigma_0\ \tau\ \sigma_1 \wedge \sigma^{\geq 1} \in X$ where the suffix $\sigma^{\geq 1}$ is η such that $\forall i \in \mathbb{N} : \eta_i = \sigma_{i+1}$.
    * $\sigma_0\ \tau\ \sigma_1$ holds since $X \subseteq \tau^{\vec{\omega}}$,
    * $\eta \in \tau^{\vec{\omega}}$ and $\forall i \in \mathbb{N} : \langle \eta_i,\ \bot\rangle = \langle \sigma_i,\ \bot\rangle \in Y$ proving that $\eta = \sigma^{\geq 1} \in X$.
- We conclude by Tarski's fixpoint transfer theorem.

□

Transfinite Iterations

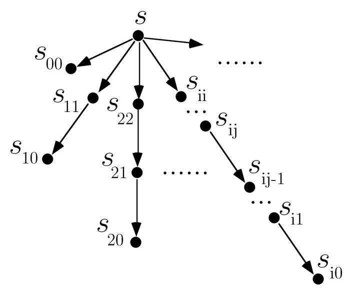
- Transition system $\langle \Sigma, \tau \rangle$:

  $\Sigma \stackrel{\Delta}{=} \{s\} \cup \{s_{ij} \mid 0 \leq j \leq i\}$
  elements of Σ are distinct two by two

  $\tau \stackrel{\Delta}{=} \{\langle s,\ s_{ii}\rangle \mid i \geq 0\} \cup \{\langle s_{ij},\ s_{ij-1}\rangle \mid 0 < j \leq i\}$

  $\tau^{\vec{\omega}} = \emptyset$

  $\tau^{\omega} = \emptyset$



- Iterates: $F^{\omega}(X) = \tau \circ X$
  - $X^0 = \{\langle s,\ \bot\rangle\} \cup \{\langle s_{ij},\ \bot\rangle \mid 0 \leq j \leq i\}$
  - $X^1 = F^{\omega}(X^0) = \{\langle s,\ \bot\rangle\} \cup \{\langle s_{ij},\ \bot\rangle \mid 1 \leq j \leq i\}$

    ...
  - $X^n = \{\langle s,\ \bot\rangle\} \cup \{\langle s_{ij},\ \bot\rangle \mid n \leq j \leq i\}$

    ...
  - $X^{\omega} = \bigcap_{n \in \mathbb{N}} X^n = \{\langle s,\ \bot\rangle\}$
  - $X^{\omega+1} = F^{\omega}(X^{\omega}) = \emptyset = \mathrm{gfp}^{\subseteq}_{\Sigma \times \{\bot\}} F^{\omega} = \tau^{\omega}$

Bifinite Relational Abstraction

- $\alpha^{\infty} \in \wp(\Sigma^{\vec{\infty}}) \longmapsto \wp(\Sigma \times \Sigma_{\bot})$, $\quad \Sigma_{\bot} \stackrel{\Delta}{=} \Sigma \cup \{\bot\}$

  $\alpha^{\infty}(X) \stackrel{\Delta}{=} \alpha^{+}(X^{\vec{+}}) \cup \alpha^{\omega}(X^{\vec{\omega}})$ where $X^{+} = X \cap (\Sigma \times \Sigma)$
  and $X^{\omega} = X \cap (\Sigma \times \{\bot\})$
- Bifinite relational semantics:

$$\begin{aligned} \tau^{\infty} &\stackrel{\Delta}{=} \alpha^{\infty}(\tau^{\vec{\infty}}) \\ &= \alpha^{+}((\tau^{\vec{\infty}})^{\vec{+}}) \cup \alpha^{\omega}((\tau^{\vec{\infty}})^{\vec{\omega}}) \\ &= \alpha^{+}(\tau^{\vec{+}}) \cup \alpha^{\omega}(\tau^{\vec{\omega}}) \\ &= \tau^{+} \cup \tau^{\omega} \end{aligned}$$

Fixpoint Bifinite Relational Semantics of a Transition System $\langle \Sigma, \tau \rangle$

- $\tau^{\infty} \stackrel{\Delta}{=} \tau^{+} \cup \tau^{\omega}$

  $= \mathrm{lfp}^{\subseteq}_{\emptyset}\ \lambda X \cdot \check{\tau} \cup \tau \circ X \ \cup\ \mathrm{lfp}^{\supseteq}_{\Sigma \times \{\bot\}}\ \lambda X \cdot \tau \circ X$

  $= \mathrm{lfp}^{\sqsubseteq^{\infty}}_{\bot^{\infty}}\ F^{\infty}$ by the bi-fixpoint theorem, where:
- $F^{\infty}(X) \stackrel{\Delta}{=} \lambda X \cdot \check{\tau} \cup \tau \circ X^{+} \cup \tau \circ X^{\omega} = \lambda X \cdot \check{\tau} \cup \tau \circ (X^{+} \cup X^{\omega})$

  $= \lambda X \cdot \check{\tau} \cup \tau \circ X$
- $X \sqsubseteq^{\infty} Y \stackrel{\Delta}{=} X^{+} \subseteq Y^{+} \ \wedge\ X^{\omega} \supseteq Y^{\omega}$
- $\bot^{\infty} \stackrel{\Delta}{=} \emptyset \cup (\Sigma \times \{\bot\}) = \Sigma \times \{\bot\}$
- $\bigsqcup^{\infty}_{i} X_i \stackrel{\Delta}{=} \bigcup_{i} X_i{}^{+} \cup \bigcap_{i} X_i{}^{\omega}$
- $\langle \wp(\Sigma \times \Sigma_{\bot}),\ \sqsubseteq^{\infty},\ \bot^{\infty},\ \sqcup^{\infty}\rangle$ is a complete lattice.

Abstraction by Parts

$$\tau^{\infty} = \alpha^{\infty}(\mathrm{lfp}_{\vec{\perp}^{\vec{\infty}}}^{\vec{\sqsubseteq}^{\vec{\infty}}} F^{\vec{\infty}}) = \mathrm{lfp}_{\perp^{\infty}}^{\sqsubseteq^{\infty}} F^{\infty}$$

- The finitary part transfers through α^{+} by Kleene's fixpoint transfer theorem (but Tarski's one is not applicable);
- The infinitary part transfers through α^{ω} by Tarski's fixpoint transfer theorem (but Kleene's one is not applicable);
- The whole transfers through α^{∞} by parts using the bifixpoint theorem (although Kleene's and Tarski's fixpoint transfer theorems are not applicable).

Denotational/Functional Nondeterministic Abstraction

We use the complete order isomorphism:

$$\langle \wp(\Sigma \times \Sigma_{\perp}),\ \sqsubseteq^{\infty},\ \perp^{\infty},\ \top^{\infty},\ \sqcup^{\infty},\ \sqcap^{\infty} \rangle \underset{\alpha^{\natural}}{\overset{\gamma^{\natural}}{\twoheadleftarrow\!\!\!\twoheadrightarrow}} \langle \Sigma \longmapsto \wp(\Sigma_{\perp}),\ \dot{\sqsubseteq}^{\natural},\ \dot{\perp}^{\natural},\ \dot{\top}^{\natural},\ \dot{\sqcup}^{\natural},\ \dot{\sqcap}^{\natural} \rangle$$

defined by the right-image of a relation:

$$\alpha^{\natural}(r) = \lambda s \bullet \{s' \in \Sigma_{\perp} \mid r(s, s')\}$$
$$\gamma^{\natural}(f) = \{\langle s,\ s' \rangle \mid s' \in f(s)\}$$

Natural Fixpoint Denotational/Functional Nondeterministic Semantics of a Transition System $\langle \Sigma,\ \tau \rangle$

- $\tau^{\natural} \stackrel{\Delta}{=} \alpha^{\natural}(\tau^{\infty})$
  $= \mathrm{lfp}_{\dot{\perp}^{\natural}}^{\dot{\sqsubseteq}^{\natural}} F^{\natural}$
- $F^{\natural}(f) \stackrel{\Delta}{=} \lambda s \bullet (\forall s' \in \Sigma : \neg(s\ \tau\ s')\ ?\ \{s\} \mid \{s' \mid \exists s'' \in \Sigma : s\ \tau\ s'' \wedge s' \in f(s'')\})$

Proof

Trivial application of Kleene's fixpoint transfert theorem for the complete order-isomorphism $\alpha^{\natural}$.

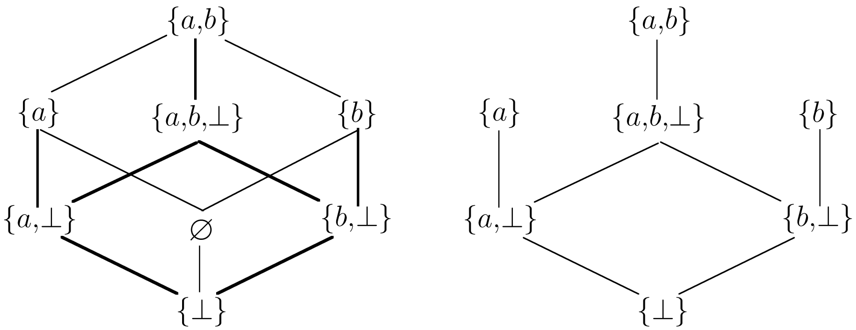
□

Computational Ordering

$$\begin{aligned}
f \dot{\sqsubseteq}^{\natural} g &\stackrel{\Delta}{=} \gamma^{\natural}(f) \sqsubseteq^{\infty} \gamma^{\natural}(g)\\
&= \{\langle s,\ s' \rangle \mid s' \in f(s) \cap \Sigma\} \subseteq \{\langle s,\ s' \rangle \mid s' \in g(s) \cap \Sigma\}\\
&\quad \wedge \{\langle s,\ s' \rangle \mid f(s) = \perp\} \subseteq \{\langle s,\ s' \rangle \mid g(s) = \perp\}\\
&= \forall s \in \Sigma : f(s)^{+} \subseteq g(s)^{+} \wedge f(s)^{\omega} \supseteq g(s)^{\omega}\\
&\qquad \text{where } X^{+} \stackrel{\Delta}{=} X \cap \Sigma \text{ and } X^{\omega} \stackrel{\Delta}{=} X \cap \{\perp\}\\
&= \forall s \in \Sigma : f(s) \sqsubseteq^{\natural} g(s)\\
&\qquad \text{where } X \sqsubseteq^{\natural} Y \stackrel{\Delta}{=} X^{+} \subseteq Y^{+} \wedge X^{\omega} \supseteq Y^{\omega}
\end{aligned}$$

This is not the classical Egli-Milner ordering!

Orderings for the Nondeterministic Denotational Semantics, $\Sigma = \{a, b\}$



Computational ordering $\sqsubseteq^\natural$ Egli-Milner ordering $\sqsubseteq^{\text{EM}}$

—— : possible iterates of $F^\natural$

Plotkin's Fixpoint Denotational/Functional Nondeterministic Semantics of a Transition System $\langle \Sigma, \tau \rangle$

- $\tau^\natural \stackrel{\Delta}{=} \alpha^\natural(\tau^\infty)$

$$= \text{lfp}^{\sqsubseteq^\natural}_{\lambda s \bullet \{\perp\}} F^\natural = \text{lfp}^{\dot\sqsubseteq^{\text{EM}}}_{\lambda s \bullet \{\perp\}} F^\natural$$

Sketch of proof

- $\text{lfp}^{\dot\sqsubseteq^{\text{EM}}}_{\lambda s \bullet \{\perp\}} F^\natural$ exists since $F^\natural$ is Egli-Milner monotonic and $\langle \wp(\Sigma_\perp) - \{\emptyset\}, \dot\sqsubseteq^{\text{EM}} \rangle$ is a cpo;
- $\text{lfp}^{\dot\sqsubseteq^{\text{EM}}}_{\lambda s \bullet \{\perp\}} F^\natural = \text{lfp}^{\dot\sqsubseteq^\natural}_{\lambda s \bullet \{\perp\}} F^\natural$ since the iterates exactly coincide.

□

Comparing the orderings $\sqsubseteq^\natural$ and $\dot\sqsubseteq^{\text{EM}}$

- The lub $\sqcup^\natural$ provides a semantics to the parallel or:

$$[\![P \mid\mid Q]\!] = [\![P]\!] \sqcup^\natural [\![Q]\!]$$

  (nontermination of $P \mid\mid Q$ only if both P and Q do not terminate);
- The lub $\sqcup^{\text{EM}}$ may not be defined.

Fixpoint Iterates Reordering

- Let $\langle\langle D, \sqsubseteq, \perp, \sqcup\rangle, F\rangle$ be a fixpoint semantic specification;
- let E be a set and $\preceq$ be a binary relation on E, such that:
  1. $\preceq$ is a pre-order on E;
  2. all iterates F^δ, $\delta \in \mathbb{O}$ of F belong to E;
  3. $\perp$ is the $\preceq$-infimum of E;
  4. the restriction $F|_E$ of F to E is $\preceq$-monotone;
  5. for all $x \in E$, if λ is a limit ordinal and $\forall \delta < \lambda : F^\delta \preceq x$ then $\bigsqcup_{\delta<\lambda} F^\delta \preceq x$.
- Then $\text{lfp}^{\sqsubseteq}_{\perp} F = \text{lfp}^{\preceq}_{\perp} F|_E \in E$.

Nondeterministic Smyth/Demoniac Denotational Semantics

- $\tau^\sharp \stackrel{\Delta}{=} \alpha^\sharp(\tau^\natural)$ where
  - $\alpha^\sharp(f) \stackrel{\Delta}{=} \lambda s \bullet f(s) \cup \{s' \in \Sigma \mid \bot \in f(s)\}$;
  - $\gamma^\sharp(g) \stackrel{\Delta}{=} g$.
- $\langle \Sigma \longmapsto \wp(\Sigma_\bot),\ \dot{\subseteq}\rangle \underset{\alpha^\sharp}{\overset{\gamma^\sharp}{\leftrightarrows}} \langle \Sigma \longmapsto (\wp(\Sigma) \cup \{\Sigma_\bot\}),\ \dot{\subseteq}\rangle$.

Demoniac Denotational Semantics in Fixpoint Form

$$\tau^\sharp = \mathrm{lfp}^{\dot{\sqsubseteq}^{=}}_{\dot{\bot}^{=}}\, F^\natural$$

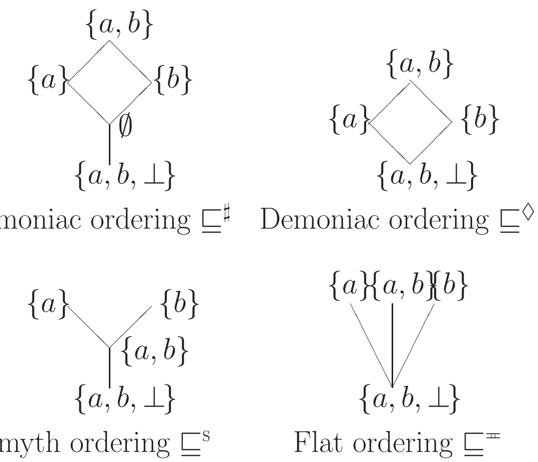
where:

- $F^\natural \stackrel{\Delta}{=} \lambda s \bullet (\forall s' \in \Sigma : \neg(s\ \tau\ s')\ ?\ \{s\}$
  $\mid \{s' \mid \exists s'' \in \Sigma : s\ \tau\ s'' \wedge s' \in f(s'')\})$
- The DCPO [20] $\langle \dot{D}^{=},\ \dot{\sqsubseteq}^{=},\ \dot{\bot}^{=},\ \dot{\sqcup}^{=}\rangle$ is the restriction of the pointwise extension of the flat DCPO $\langle D^{=},\ \sqsubseteq^{=},\ \bot^{=},\ \sqcup^{=}\rangle$;
- $D^{=} \stackrel{\Delta}{=} (\wp(\Sigma) \setminus \{\emptyset\}) \cup \{\bot^{=}\}$
- $\bot^{=} \stackrel{\Delta}{=} \Sigma_\bot$
- $\dot{D}^{=} \stackrel{\Delta}{=} \{f \in \Sigma \longmapsto D^{=} \mid \forall s, s' \in \Sigma : (s' \in f(s) \wedge f(s) \neq \bot^{=}) \Rightarrow$
  $(s' \in \check{\tau} \wedge f(s') = \{s'\})\}$.

This is not the classical Smyth ordering!

[20] Directed Complete POset.

Examples of Other Possible Demoniac Iterate Orderings



Minimality of $\langle \dot{D}^{=},\ \dot{\sqsubseteq}^{=}\rangle$

- Let $\langle E,\ \preccurlyeq\rangle$ be any poset such that:
  - $\dot{\bot}^{=}$ is the $\preccurlyeq$-infimum of E,
  - $F^\natural[\![\tau]\!] \stackrel{\Delta}{=} \lambda s \bullet (\forall s' \in \Sigma : \neg(s\ \tau\ s')\ ?\ \{s\} \mid \{s' \mid \exists s'' \in \Sigma : s\ \tau\ s'' \wedge s' \in f(s'')\}) \in E \overset{m}{\longmapsto} E$ is $\preccurlyeq$-monotone, and
  - $\forall \tau : \tau^\sharp = \mathrm{lfp}^{\preccurlyeq}_{\dot{\bot}^{=}}\, F^\natural[\![\tau]\!]$

  then:
  - $\dot{D}^{=} \subseteq E$, and
  - $\dot{\sqsubseteq}^{=} \subseteq \preccurlyeq$.

Hoare/Angelic Denotational Semantics

- $\tau^\flat \stackrel{\Delta}{=} \dot{\alpha}^\flat(\tau^\natural)$
- $\dot{\alpha}^\flat(\varphi) \stackrel{\Delta}{=} \lambda s \cdot \varphi(s) \cap \Sigma$
- $\dot{\gamma}^\flat(\phi) \stackrel{\Delta}{=} \lambda s \cdot \phi(s) \cup \{\bot\}$
- $\langle \Sigma \longmapsto \wp(\Sigma_\bot),\ \dot{\subseteq} \rangle \underset{\dot{\alpha}^\flat}{\overset{\dot{\gamma}^\flat}{\longleftarrow\!\!\!\longrightarrow}} \langle \Sigma \longmapsto \wp(\Sigma),\ \dot{\subseteq} \rangle$
- $\tau^\flat = \mathrm{lfp}^{\dot{\subseteq}}_{\dot{\emptyset}}\ F^\natural$ where $F^\natural = \lambda s \cdot (\forall s' \in \Sigma : \neg(s\ \tau\ s')\ ?\ \{s\} \mid \{s' \mid \exists s'' \in \Sigma : s\ \tau\ s'' \wedge s' \in f(s'')\})$ is a complete $\dot{\cup}$-morphism on the complete lattice $\langle \Sigma \longmapsto \wp(\Sigma),\ \dot{\subseteq},\ \dot{\emptyset},\ \lambda s \cdot \Sigma,\ \dot{\cup},\ \dot{\cap} \rangle$ which is the pointwise extension of the powerset $\langle \wp(\Sigma),\ \emptyset \rangle$.

Denotational/Functional Deterministic Semantics

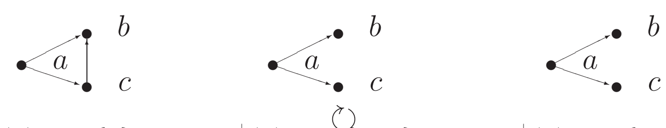
Denotational/Functional Deterministic Abstraction

- $\langle \wp(\Sigma_\bot),\ \subseteq \rangle \underset{\alpha^s}{\overset{\gamma^s}{\longleftarrow\!\!\!\longrightarrow}} \langle \Sigma \cup \{\bot, \top\},\ \sqsubseteq^s \rangle$ where $\forall s \in \Sigma : \bot \sqsubseteq^s \bot \sqsubseteq^s s \sqsubseteq^s s \sqsubseteq^s \top \sqsubseteq^s \top$
- The abstraction α^s disregards nondeterminism:

$$\alpha^s(\emptyset) \stackrel{\Delta}{=} \bot \qquad \gamma^s(\bot) \stackrel{\Delta}{=} \{\bot\}$$
$$\alpha^s(\{\bot\}) \stackrel{\Delta}{=} \bot$$
$$\alpha^s(\{s\}) = \alpha^s(\{s, \bot\}) \stackrel{\Delta}{=} s,\ s \in \Sigma \qquad \gamma^s(s) \stackrel{\Delta}{=} \{s, \bot\}$$
$$\alpha^s(X) \stackrel{\Delta}{=} \top,\ \text{otherwise} \qquad \gamma^s(\top) \stackrel{\Delta}{=} \Sigma_\bot$$

- $\langle \Sigma \longmapsto \wp(\Sigma_\bot),\ \subseteq \rangle \underset{\dot{\alpha}^s}{\overset{\dot{\gamma}^s}{\longleftarrow\!\!\!\longrightarrow}} \langle \Sigma \longmapsto (\Sigma \cup \{\bot, \top\}),\ \dot{\sqsubseteq}^s \rangle$ where $\dot{\alpha}^s(f) \stackrel{\Delta}{=} \lambda s \cdot \alpha^s(f(s))$ and $\dot{\gamma}^s(f) \stackrel{\Delta}{=} \lambda s \cdot \gamma^s(f(s))$

Natural $\tau^\natural$ and deterministic $\tau^\top$ denotational semantics of nondeterministic transition systems τ



$\tau^\natural(a) = \{b\}$, $\tau^\top(a) = b$, $\tau^\top(b) = b$, $\tau^\top(c) = b$

$\tau^\natural(a) = \{b, \bot\}$, $\tau^\top(a) = b$, $\tau^\top(b) = b$, $\tau^\top(c) = \bot$

$\tau^\natural(a) = \{b, c\}$, $\tau^\top(a) = \top$, $\tau^\top(b) = b$, $\tau^\top(c) = c$

Fixpoint Denotational/Functional Deterministic Semantics of a Transition System $\langle \Sigma,\ \tau \rangle$

- $\tau^s \stackrel{\Delta}{=} \dot{\alpha}^s(\tau^\natural) = \dot{\alpha}^s(\mathrm{lfp}^{\sqsubseteq^\natural}_{\lambda s \bullet \{\perp\}} F^\natural) = \mathrm{lfp}^{\sqsubseteq^s}_{\lambda s \bullet \perp} F^s$
- $F^s \stackrel{\Delta}{=} \lambda f \bullet \lambda s \bullet (\forall s' \in \Sigma : \neg(s\ \tau\ s')\ ?\ s \mid \sqcup^s \{f(s'') \mid s\ \tau\ s'\})$

Proof

- $\dot{\alpha}^s(\lambda s \bullet \{\perp\}) = \lambda s \bullet \perp$;
- $\dot{\alpha}^s \circ F^d = F^s \circ \dot{\alpha}^s$ leads to the definition of F^d;
- $\dot{\alpha}^s(\dot{\bigsqcup}_i^\infty f_i) = \dot{\bigsqcup}_i^{ss} \dot{\alpha}^s(f_i)$ leads to the definition of the $\sqsubseteq^s$-lub $\dot{\sqcup}^s$;
- F^s is monotonic for $\sqsubseteq^s$;
- Kleene's fixpoint transfer theorem applies.

$\square$

Deterministic Transition System, Scott's Semantics

- If τ is deterministic, then $\tau \in \Sigma \not\mapsto \Sigma$ and

$$F^s = \lambda f \bullet \lambda s \bullet (s \notin \mathrm{dom}\,\tau\ ?\ s \mid \tau(s)) \qquad (1)$$

- $\top$ is unreachable and can be eliminated from the domain so that $\sqsubseteq^s$ is exactly Scott ordering.

The Rôle of $\top$

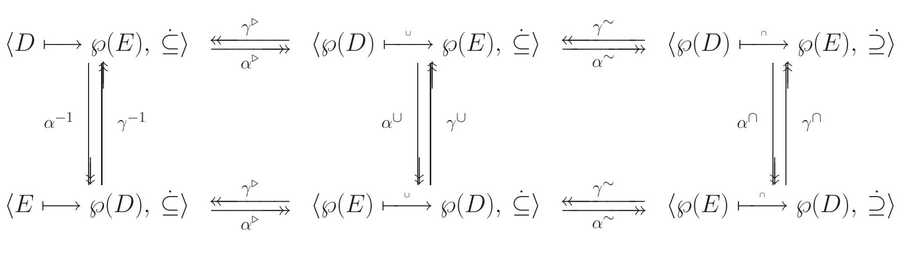
- The top element $\top$ is often eliminated from Scott's domains by lack of intuitive interpretation;
- We interpret $\top$ as an abstraction forgetting about nondeterminism.

Predicate Transformer Semantics

Nondeterministic Denotational to Predicate Transformer Abstractions

$$\alpha^{-1} \triangleq \lambda f \in D \longmapsto \wp(E)\bullet\, \lambda s'\bullet\{s \mid s' \in f(s)\}$$
$$\gamma^{-1} \triangleq \lambda f \in E \longmapsto \wp(D)\bullet\, \lambda s\bullet\{s' \mid s \in f(s')\}$$
$$\alpha^{\triangleright} \triangleq \lambda f \in D \longmapsto \wp(E)\bullet\, \lambda P \in \wp(D)\bullet\{s' \mid \exists s \in P : s' \in f(s)\}$$
$$\gamma^{\triangleright} \triangleq \lambda \Psi \in \wp(D) \stackrel{\cup}{\longmapsto} \wp(E)\bullet\, \lambda s\bullet\, \Psi(\{s\})$$
$$\alpha^{\cup} \triangleq \lambda \Psi \in \wp(D) \stackrel{\cup}{\longmapsto} \wp(E)\bullet\, \lambda Q \in \wp(E)\bullet\{s \mid \Psi(\{s\}) \cap Q \neq \emptyset\}$$
$$\gamma^{\cup} \triangleq \lambda \Psi \in \wp(E) \stackrel{\cup}{\longmapsto} \wp(D)\bullet\, \lambda P \in \wp(D)\bullet\{s' \mid \Psi(\{s'\}) \cap P \neq \emptyset\}$$
$$\alpha^{\sim} \triangleq \lambda \Psi \in \wp(D) \stackrel{\cup}{\longmapsto} \wp(E)\bullet\, \lambda P \in \wp(D)\bullet\, \neg(\Psi(\neg P))$$
$$\gamma^{\sim} \triangleq \lambda \Psi \in \wp(E) \stackrel{\cap}{\longmapsto} \wp(D)\bullet\, \lambda P \in \wp(D)\bullet\, \neg(\Psi(\neg P))$$
$$\alpha^{\cap} \triangleq \lambda \Phi \in \wp(D) \stackrel{\cap}{\longmapsto} \wp(E)\bullet\, \lambda Q \in \wp(E)\bullet\{s \mid \Phi(\neg\{s\}) \cup Q = E\}$$
$$\gamma^{\cap} \triangleq \lambda \Phi \in \wp(E) \stackrel{\cap}{\longmapsto} \wp(D)\bullet\, \lambda P \in \wp(D)\bullet\{s' \mid \Phi(\neg\{s'\}) \cup P = D\}$$

Galois Connection Commutative Diagram



Predicate Transformer Abstractions

If $f \in D \longmapsto \wp(E)$):

$$\begin{aligned}
\mathrm{gsp}\llbracket f\rrbracket &\triangleq \alpha^{\triangleright}[f] \;\in\; \wp(D) \stackrel{\cup}{\longmapsto} \wp(E)\\
&= \lambda P \in \wp(D)\bullet\{s' \in E \mid \exists s \in P : s' \in f(s)\}\\
\mathrm{gspa}\llbracket f\rrbracket &\triangleq \alpha^{\sim} \circ \alpha^{\triangleright}[f] \;\in\; \wp(D) \stackrel{\cap}{\longmapsto} \wp(E)\\
&= \lambda P \in \wp(D)\bullet\{s' \in E \mid \forall s \in D : s' \in f(s) \Rightarrow s \in P\}\\
\mathrm{gwp}\llbracket f\rrbracket &\triangleq \alpha^{\sim} \circ \alpha^{\triangleright} \circ \alpha^{-1}[f] \;\in\; \wp(E) \stackrel{\cap}{\longmapsto} \wp(D)\\
&= \lambda Q \in \wp(E)\bullet\{s \in D \mid \forall s' \in E : s' \in f(s) \Rightarrow s' \in Q\}\\
\mathrm{gwpa}\llbracket f\rrbracket &\triangleq \alpha^{\triangleright} \circ \alpha^{-1}[f] \;\in\; \wp(E) \stackrel{\cup}{\longmapsto} \wp(D)\\
&= \lambda Q \in \wp(E)\bullet\{s \in D \mid \exists s' \in Q : s' \in f(s)\}
\end{aligned}$$

Generalized Weakest Precondition Semantics

- $\tau^{\mathrm{gwp}} \triangleq \mathrm{gwp}\llbracket \tau^{\natural}\rrbracket = \mathrm{lfp}^{\sqsubseteq^{\mathrm{gwp}}}_{\bot^{\mathrm{gwp}}}\, F^{\mathrm{gwp}}$
- $F^{\mathrm{gwp}} \in D^{\mathrm{gwp}} \stackrel{m}{\longmapsto} D^{\mathrm{gwp}} \triangleq \lambda\Phi\bullet\, \lambda Q\bullet(\neg\check{\tau} \cup Q)\, \dot{\cap}\, \mathrm{gwp}\llbracket \tau^{\blacktriangleright}\rrbracket \circ \Phi$
  $= \lambda\Phi\bullet\, \lambda Q\bullet(Q \cap \check{\tau})\, \dot{\cup}\, \mathrm{wp}\llbracket \tau^{\blacktriangleright}\rrbracket \circ \Phi$
  is a $\sqsubseteq^{\mathrm{gwp}}$-monotone map on the complete lattice $\langle D^{\mathrm{gwp}},\ \sqsubseteq^{\mathrm{gwp}},\ \bot^{\mathrm{gwp}},\ \sqcup^{\mathrm{gwp}}\rangle$
- $\mathrm{wp}\llbracket f\rrbracket\, Q \triangleq \{s \in \Sigma \mid \exists s' \in \Sigma : s' \in f(s) \wedge \forall s' \in f(s) : s' \in Q\}$
- $D^{\mathrm{gwp}} \triangleq \wp(\Sigma_{\bot}) \stackrel{\cap}{\longmapsto} \wp(\Sigma)$,
- $\Phi \sqsubseteq^{\mathrm{gwp}} \Psi \triangleq \forall Q \subseteq \Sigma : \Psi(Q \cup \{\bot\}) \subseteq \Phi(Q \cup \{\bot\}) \wedge \Phi(\Sigma) \subseteq \Psi(\Sigma)$,
- $\bot^{\mathrm{gwp}} = \lambda Q\bullet(\bot \in Q\ ?\ \Sigma \mid \emptyset)$
- $\bigsqcup^{\mathrm{gwp}}_{i\in\Delta} \Psi_i \triangleq \lambda Q\bullet \bigcap_{i\in\Delta} \Psi_i(Q \cup \{\bot\}) \cap (\bot \notin Q\ ?\ \bigcup_{i\in\Delta} \Psi_i(\Sigma) \mid \Sigma)$.

Dijkstra's Weakest Conservative Precondition Abstraction

- $\langle D^{gwp},\ \dot{\supseteq}\rangle \overset{\gamma^{wp}}{\underset{\alpha^{wp}}{\leftrightarrows}} \langle D^{wp},\ \dot{\supseteq}\rangle$ where $D^{wp} \stackrel{\Delta}{=} \wp(\Sigma) \overset{\cap}{\longmapsto} \wp(\Sigma)$, $\alpha^{wp} \stackrel{\Delta}{=} \lambda\Phi\bullet\, \Phi|_{\wp(\Sigma)}$ and $\gamma^{wp}(\Psi) \stackrel{\Delta}{=} \lambda Q\bullet(\bot \notin Q\ ?\ \Psi(Q) \mid \emptyset)$;
- $\tau^{wp} \stackrel{\Delta}{=} \alpha^{wp}(\tau^{gwp}) = \alpha^{wp}(\mathrm{gwp}[\![\tau^{\sharp}]\!])$;
- Dikstra's fixpoint characterization of τ^{wp} is for a given postcondition Q;
- If $Q \subseteq E$ then $\langle \wp(E) \overset{\cap}{\longmapsto} \wp(D),\ \dot{\supseteq}\rangle \overset{\gamma^{Q}}{\underset{\alpha^{Q}}{\leftrightarrows}} \langle \wp(D),\ \supseteq\rangle$ where $\alpha^{Q}(\Phi) \stackrel{\Delta}{=} \Phi(Q)$ and $\gamma^{Q}(P) \stackrel{\Delta}{=} \lambda R\bullet(Q \subseteq R\ ?\ P \mid \emptyset)$;

Dijkstra's Weakest Conservative Precondition Semantics

From $\tau^{wp}(Q) = \alpha^{Q}(\alpha^{wp}(\tau^{gwp}))$ and Kleene fixpoint transfer theorem, we derive:

- $\tau^{wp}(Q) = \lambda Q\bullet\, \mathrm{lfp}^{\subseteq}_{\emptyset}\, F^{wp}[\![Q]\!]$
- $F^{wp} \in \wp(\Sigma) \longmapsto \wp(\Sigma) \overset{m}{\longmapsto} \wp(\Sigma)$
- $\tau^{\blacktriangleright}(s) \stackrel{\Delta}{=} \{s' \mid s\ \tau\ s'\}$;
- $F^{wp}[\![Q]\!] \stackrel{\Delta}{=} \lambda P\bullet(Q \cap \check{\tau}) \cup \mathrm{wp}[\![\tau^{\blacktriangleright}]\!]\, P$
  $= \lambda P\bullet(\neg\check{\tau} \cup Q) \cap \mathrm{gwp}[\![\tau^{\blacktriangleright}]\!]\, P$
  is a $\subseteq$-monotone map on the complete lattice $\langle \wp(\Sigma),\ \subseteq,\ \emptyset,\ \Sigma,\ \cup,\ \cap\rangle$.

Dijkstra's Weakest Liberal Precondition Semantics

- $\langle D^{gwp},\ \dot{\supseteq}\rangle \overset{\gamma^{wlp}}{\underset{\alpha^{wlp}}{\leftrightarrows}} \langle D^{wlp},\ \dot{\supseteq}\rangle$ where $D^{wlp} \stackrel{\Delta}{=} \wp(\Sigma) \overset{\cap}{\longmapsto} \wp(\Sigma)$, $\alpha^{wlp} \stackrel{\Delta}{=} \lambda\Phi\bullet\, \lambda Q\bullet\, \Phi(Q \cup \{\bot\})$ and $\gamma^{wlp}(\Psi) \stackrel{\Delta}{=} \lambda Q\bullet(\bot \in Q\ ?\ \Psi(Q) \mid \emptyset)$;
- $\tau^{wlp} \stackrel{\Delta}{=} \alpha^{wlp}(\tau^{gwp}) = \mathrm{gwp}[\![\tau^{\flat}]\!]$;
- By Kleene fixpoint transfer, $\tau^{wlp} = \lambda Q\bullet\, \mathrm{gfp}^{\subseteq}_{\Sigma}\, F^{wp}[\![Q]\!]$.

Correspondence Between Pre- and Postcondition Semantics

If $f \in D \longmapsto \wp(E)$ then $\langle \wp(D),\ \subseteq\rangle \overset{\mathrm{gwp}[\![f]\!]}{\underset{\mathrm{gsp}[\![f]\!]}{\leftrightarrows}} \langle \wp(E),\ \subseteq\rangle$.

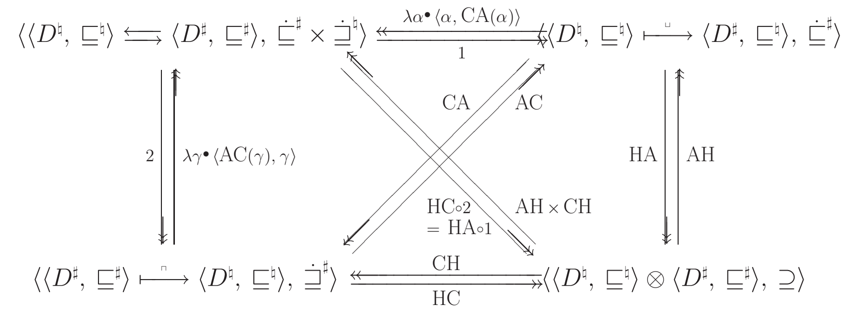
Axiomatic Semantics

Galois Connections, Complete Join/Meet Morphisms and Tensor Product

- G. c.: $\langle D^\natural, \sqsubseteq^\natural\rangle \leftrightarrows \langle D^\sharp, \sqsubseteq^\sharp\rangle \triangleq \{\langle\alpha, \gamma\rangle \mid \langle D^\natural, \sqsubseteq^\natural\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle D^\sharp, \sqsubseteq^\sharp\rangle\}$;
- Complete join morphisms: $D^\natural \overset{\sqcup}{\longmapsto} D^\sharp \triangleq \{\alpha \in D^\natural \longmapsto D^\sharp \mid \forall X \subseteq D^\natural : \alpha(\sqcup^\natural X) = \sqcup^\sharp \alpha^\blacktriangleright(X)\}$;
- Complete meet morphisms: $D^\sharp \overset{\sqcap}{\longmapsto} D^\natural \triangleq \{\gamma \in D^\sharp \longmapsto D^\natural \mid \forall Y \subseteq D^\sharp : \gamma(\sqcap^\sharp Y) = \sqcap^\natural \gamma^\blacktriangleright(Y)\}$;
- Tensor products: $\langle D^\natural, \sqsubseteq^\natural\rangle \otimes \langle D^\sharp, \sqsubseteq^\sharp\rangle \triangleq \{H \in \wp(D^\natural \times D^\sharp) \mid (1) \wedge (2) \wedge (3)\}$ where the conditions are:
  1. $(X \sqsubseteq^\natural X' \wedge \langle X', Y'\rangle \in H \wedge Y' \sqsubseteq^\sharp Y) \Rightarrow (\langle X, Y\rangle \in H)$;
  2. $(\forall i \in \Delta : \langle X_i, Y\rangle \in H) \Rightarrow (\langle \bigsqcup^\natural_{i\in\Delta} X_i, Y\rangle \in H)$;
  3. $(\forall i \in \Delta : \langle X, Y_i\rangle \in H) \Rightarrow (\langle X, \bigsqcap^\natural_{i\in\Delta} Y_i\rangle \in H)$.

Galois Connection Commutative Diagram

$$1(\langle\alpha, \gamma\rangle) \triangleq \alpha \qquad \mathrm{HA}(\alpha) \triangleq \{\langle x, y\rangle \in D^\natural \times D^\sharp \mid \alpha(x) \sqsubseteq^\sharp y\}$$
$$2(\langle\alpha, \gamma\rangle) \triangleq \gamma \qquad \mathrm{HC}(\gamma) \triangleq \{\langle x, y\rangle \in D^\natural \times D^\sharp \mid x \sqsubseteq^\natural \gamma(y)\}$$
$$\mathrm{AC}(\gamma) \triangleq \lambda x \bullet \sqcap^\sharp\{y \mid x \sqsubseteq^\natural \gamma(y)\} \qquad \mathrm{AH}(H) \triangleq \lambda x \bullet \sqcap^\sharp\{y \mid \langle x, y\rangle \in H\}$$
$$\mathrm{CA}(\alpha) \triangleq \lambda y \bullet \sqcup^\natural\{x \mid \alpha(x) \sqsubseteq^\sharp y\} \qquad \mathrm{CH}(H) \triangleq \lambda y \bullet \sqcup^\natural\{x \mid \langle x, y\rangle \in H\}$$



Floyd/Hoare/Naur Partial Correctness Semantics

- $\tau^{\mathrm{pH}} \triangleq \mathrm{HC}(\tau^{\mathrm{wlp}})$;
- $\tau^{\mathrm{pH}} = \{\langle P, Q\rangle \in \wp(\Sigma) \otimes \wp(\Sigma) \mid \exists I \in \wp(\Sigma) : P \subseteq I \wedge I \subseteq \mathrm{gwp}[\![\tau^\blacktriangleright]\!] I \wedge (I \cap \check{\tau}) \subseteq Q\}$.

Proof By Park fixpoint induction: if $\langle D, \sqsubseteq, \bot, \top, \sqcup, \sqcap\rangle$ is a complete lattice, $F \in D \overset{\mathrm{m}}{\longmapsto} D$ is $\sqsubseteq$-monotone and $L \in D$ then $\mathrm{lfp}^{\sqsubseteq}_{\bot} F \sqsubseteq P \iff (\exists I : F(I) \sqsubseteq I \wedge I \sqsubseteq P)$. □

Hoare Logic

- Hoare triples: $\{P\}\tau^{\vec{\infty}}\{Q\} \stackrel{\Delta}{=} \langle P,\ Q\rangle \in \tau^{\mathrm{pH}}$, $\{P\}\tau\{Q\} \stackrel{\Delta}{=} P \subseteq \mathrm{gwp}[\![\tau^{\blacktriangleright}]\!]\, Q$;
- Hoare logic: $\{P\}\tau^{\vec{\infty}}\{Q\}$ if and only if it derives from the axiom:

$$\{\mathrm{gwp}[\![\tau^{\blacktriangleright}]\!]\, Q\}\tau\{Q\} \qquad (\tau)$$

and the following inference rules:

$$\frac{P \subseteq P',\ \{P'\}\tau^{\vec{\infty}}\{Q'\},\ Q' \subseteq Q}{\{P\}\tau^{\vec{\infty}}\{Q\}}\ (\Rightarrow) \qquad \frac{\{P_i\}\tau^{\vec{\infty}}\{Q\},\ i \in \Delta}{\{\bigcup_{i\in\Delta} P_i\}\tau^{\vec{\infty}}\{Q\}}\ (\vee)$$

$$\frac{\{P\}\tau^{\vec{\infty}}\{Q_i\},\ i \in \Delta}{\{P\}\tau^{\vec{\infty}}\{\bigcap_{i\in\Delta} Q_i\}}\ (\wedge) \qquad \frac{\{I\}\tau\{I\}}{\{I\}\tau^{\vec{\infty}}\{I \cap \check{\tau}\}}\ (\tau^{\vec{\infty}})$$

Floyd Total Correctness Semantics

- $\tau^{\mathrm{tH}} \stackrel{\Delta}{=} \mathrm{HC}(\tau^{\mathrm{wp}})$;
- $\tau^{\mathrm{tH}} = \{\langle P,\ Q\rangle \in \wp(\Sigma) \otimes \wp(\Sigma) \mid \exists \epsilon \in \mathbb{O} : \exists I \in (\epsilon + 1) \longmapsto \wp(\Sigma) :$
  $\forall \delta \leq \epsilon : I^\delta \subseteq (\neg\check{\tau} \cup Q) \cap \mathrm{gwp}[\![\tau^{\blacktriangleright}]\!]\big(\bigcup_{\beta<\delta} I^\beta\big) \wedge P \subseteq I^\epsilon\}$.
- Floyd (equivalent) verification conditions:

$$\forall s \in I^\delta : \bigvee \begin{array}{l} \forall s' : \neg(s\ \tau\ s') \wedge s \in Q \\ \exists s' : s\ \tau\ s' \wedge \forall s' : s\ \tau\ s' \Rightarrow (\exists \beta < \delta : s' \in I^\beta) \end{array}$$

Proof By the lower fixpoint induction principle: if $\langle D,\ \sqsubseteq,\ \bot,\ \sqcup\rangle$ is a DCPO, $F \in D \stackrel{\mathrm{m}}{\longmapsto} D$ is $\sqsubseteq$-monotone, $\bot \in D$ satisfies $\bot \sqsubseteq F(\bot)$ and $P \in D$ then $P \sqsubseteq \mathrm{lfp}_{\bot}^{\sqsubseteq} F \iff (\exists \epsilon \in \mathbb{O} : \exists I \in (\epsilon + 1) \longmapsto D : I^0 \sqsubseteq \bot \wedge \forall \delta : 0 < \delta \leq \epsilon \Rightarrow I^\delta \sqsubseteq F(\bigsqcup_{\zeta<\delta} I^\zeta) \wedge P \sqsubseteq I^\epsilon)$. $\square$

Manna/Pnueli Total Correctness Logic

- Manna/Pnueli triples: $[P]\tau^{\vec{\infty}}[Q] \stackrel{\Delta}{=} \langle P,\ Q\rangle \in \tau^{\mathrm{tH}}$, $[P]\tau[Q] \stackrel{\Delta}{=} P \subseteq \mathrm{gwp}[\![\tau^{\blacktriangleright}]\!]\, Q$;
- Manna/Pnueli total correctness axiomatic semantics: $[P]\tau^{\vec{\infty}}[Q]$ if and only if it derives from the axiom (τ), the inference rules $(\Rightarrow)$, $(\wedge)$, $(\vee)$ and the following:

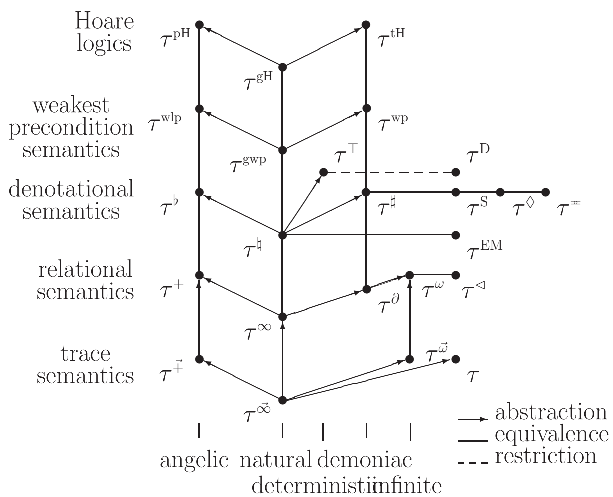
$$\frac{I^0 \subseteq Q \cap \check{\tau},\quad \bigwedge_{\delta=1}^{\epsilon} I^\delta \subseteq \neg\check{\tau} \cup Q,\quad \bigwedge_{\delta=1}^{\epsilon} [I^\delta]\tau[\bigcup_{\beta<\delta} I^\beta]}{[I^\epsilon]\tau^{\vec{\infty}}[Q]}\ (\tau^{\vec{\infty}})$$

Lattice of Semantics

Comparison of Semantics

- $\tau^\natural \in D^\natural \leq \tau^\sharp \in D^\sharp$ iff $\tau^\sharp = \alpha^\sharp(\tau^\natural)$ and $\langle D^\natural, \leq\rangle \xrightleftharpoons[\alpha^\sharp]{\gamma^\sharp} \langle D^\sharp, \leqq\rangle$ is a preorder between semantics;
- The quotient poset is isomorphic to Ward lattice of upper closure operators $\gamma^\sharp \circ \alpha^\sharp$ on $\langle D^{\vec{\infty}}, \subseteq\rangle$;
- We get a lattice of semantics which is part of the lattice of abstract interpretations.

Lattice of Semantics



Application to the (Eager) Lambda-Calculus (Prospective)

Relational Semantics with Closures

$$E \vdash \lambda \mathtt{x} \cdot e \Rightarrow \langle \mathtt{x},\ e,\ E\rangle \qquad \frac{E \vdash e_1 \Rightarrow \bot}{E \vdash e_1(e_2) \Rightarrow \bot}$$

$$\frac{c = \langle \mathtt{x},\ e,\ E[f \leftarrow c]\rangle}{E \vdash \mu \mathtt{f} \cdot \lambda \mathtt{x} \cdot e \Rightarrow c} \qquad \frac{E \vdash e_1 \Rightarrow \langle \mathtt{x}',\ e',\ E'\rangle \quad E \vdash e_2 \Rightarrow \bot}{E \vdash e_1(e_2) \Rightarrow \bot}$$

$$\frac{E \vdash e_1 \Rightarrow \langle \mathtt{x}',\ e',\ E'\rangle \quad E \vdash e_2 \Rightarrow v, v \neq \Omega \quad E'[\mathtt{x}' \leftarrow v] \vdash e' \Rightarrow r}{E \vdash e_1(e_2) \Rightarrow r} \qquad \frac{E \vdash e_1 \Rightarrow \langle \mathtt{x}',\ e',\ E'\rangle \quad E \vdash e_2 \Rightarrow v, v \neq \Omega \quad E'[\mathtt{x}' \leftarrow v] \vdash e' \Rightarrow \bot}{E \vdash e_1(e_2) \Rightarrow \bot}$$

Denotational Semantics

$$\begin{array}{rcll} \mathsf{u}, \mathsf{f}, \varphi \in \mathbb{U} & \cong & \{\Omega\}_\perp^\top \oplus \mathbb{Z}_\perp^\top \oplus [\mathbb{U} \longmapsto \mathbb{U}]_\perp^\top & \text{values} \\ \mathsf{R} \in \mathbb{R} & \stackrel{\Delta}{=} & \mathbb{X} \longmapsto \mathbb{U} & \text{environments} \\ \phi \in \mathbb{S} & \stackrel{\Delta}{=} & \mathbb{R} \longmapsto \mathbb{U} & \text{semantic domain} \end{array}$$

$$\begin{array}{rcl} \mathbf{S}[\![\lambda \mathtt{x} \cdot e]\!]\mathsf{R} & \stackrel{\Delta}{=} & \lambda \mathsf{u} \cdot (\mathsf{u} = \perp \,?\, \perp \\ & & \quad \mid \mathsf{u} = \Omega \,?\, \Omega \\ & & \quad \mid \mathbf{S}[\![e]\!]\mathsf{R}[\mathtt{x} \leftarrow \mathsf{u}]) \\ \mathbf{S}[\![e_1(e_2)]\!]\mathsf{R} & \stackrel{\Delta}{=} & (\mathbf{S}[\![e_1]\!]\mathsf{R} = \perp \vee \mathbf{S}[\![e_2]\!]\mathsf{R} = \perp \,?\, \perp \\ & & \mid \mathbf{S}[\![e_1]\!]\mathsf{R} = \mathsf{f} \in [\mathbb{U} \longmapsto \mathbb{U}] \,?\, \mathsf{f}\big(\mathbf{S}[\![e_2]\!]\mathsf{R}\big) \\ & & \mid \Omega) \\ \mathbf{S}[\![\mu \mathtt{f} \cdot \lambda \mathtt{x} \cdot e]\!]\mathsf{R} & \stackrel{\Delta}{=} & \mathrm{lfp}^{\sqsubseteq}\, \lambda \varphi \cdot \mathbf{S}[\![\lambda \mathtt{x} \cdot e]\!]\mathsf{R}[\mathtt{f} \leftarrow \varphi] \end{array}$$

Abstraction

- The rules of the relational semantics can be interpreted as least fixpoints for the bifinite ordering;
- The abstraction function $\alpha \in \wp(\mathbb{V}) \longmapsto \mathbb{U}$ is as follows [21]:

$$\begin{array}{rcl} \alpha(\emptyset) & \stackrel{\Delta}{=} & \perp \\ \alpha(\{\perp\}) & \stackrel{\Delta}{=} & \perp \\ \alpha(\{z\}) \;=\; \alpha(\{z, \perp\}) & \stackrel{\Delta}{=} & z, \quad z \in \mathbb{Z} \\ \alpha(\{\Omega\}) & \stackrel{\Delta}{=} & \Omega \\ \alpha(X) & \stackrel{\Delta}{=} & \top, \quad \text{otherwise.} \\ \alpha(\langle \mathtt{x},\, e,\, E \rangle) & \stackrel{\Delta}{=} & \lambda \mathsf{u} \in \mathbb{U} \cdot \alpha(\{r \mid \exists v \in \mathbb{V} : \alpha(\{v\}) = \mathsf{u} \wedge \\ & & \qquad E[\mathtt{x} \leftarrow v] \vdash\!\!- e \Rightarrow r\}) \end{array}$$

[21] Liftinga and injections are omitted.

- $\alpha \in (\mathbb{X} \longmapsto \mathbb{V}) \longmapsto (\mathbb{X} \longmapsto \mathbb{U})$:

$$\alpha(E) \stackrel{\Delta}{=} \lambda \mathtt{x} \cdot \alpha(E(\mathtt{x}))$$

- $\alpha \in \wp((\mathbb{X} \longmapsto \mathbb{V}) \times \mathbb{V}) \longmapsto ((\mathbb{X} \longmapsto \mathbb{U}) \longmapsto \mathbb{U})$:

$$\alpha(\Phi[\![e]\!]) \stackrel{\Delta}{=} \lambda \mathsf{R} \cdot \alpha(\{r \mid \exists E : \alpha(E) = \mathsf{R} \wedge E \vdash\!\!- e \Rightarrow r \in \Phi[\![e]\!]\})$$

Alternative Partitionning of Executions

- We have explored linear time (set of traces) semantics with partition between finite and infinite traces;
- A different partitionning for branching time (tree) semantics would be states with or without later possibility to branch toward a nonterminating execution.

Need for semantics at various levels of refinement

- Many semantics at different levels of abstraction are needed for program analysis;
- A unified framework for presenting all these semantics seems indispensable.

Further Work for Semanticians

- Consider realistic practical languages (C^{++}, Java, ML, etc);
- Consider computable approximations of semantic domains (to be used in program analysis);
- A need for mathematical foundations but also applications of programming semantics;
- A lot of work for future applied semanticians (like applied mathematicians).