Abstract Interpretation of Graphs

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Dedicated to Manuel Hermenegildo
for his 60th birthday and many years of friendship

Abstract. Path problems in graphs can be solved by abstraction of a fixpoint definition of all paths in a finite graph. Applied to the Roy-Floyd-Warshall shortest path algorithm this yields a naïve n^4 algorithm where n is the number of graph vertices. By over-approximating the elementary paths and cycles and generalizing the classical exact fixpoint abstraction, we constructively derive the classical n^3 Roy-Floyd-Warshall algorithm.

1 Introduction

1.1 Objectives

[2,9,11,14,15] observed that various graph path algorithms can be designed and proved correct based on a common algebraic structure and then instantiated to various path problems up to homomorphisms. We show that this structure originates from the fixpoint characterization of the set of graph paths using the set of graph edges, the concatenation and union of sets of paths as basic operations. The common algebraic structure of graph path algorithms follows from the fact that these primitives and the fixpoint are preserved by abstraction with Galois connections. For example [19] designs Bellman–Ford–Moore algorithm [1, Sect. 2.3.4] by abstraction of a fixpoint definition of all graph paths (where a path is a vertex or a path concatenated with an arc).

The same approach for the Roy-Floyd-Warshall algorithm [1, Sect. 2.3.5], [12, p. 26–29], [13], and [18, p. 129] (where a path is an arc or the concatenation of a path with a path) yields a naïve algorithm in $O(n^4)$ where n is the number of vertices of the weighted finite graph (assumed to have no cycle of strictly negative weight). The derivation of the original Roy-Floyd-Warshall algorithm in $O(n^3)$ is tricky since it is based on the abstraction of an over-approximation of the elementary paths which is an under-approximation of all graph paths. It requires a generalization of the classical complete fixpoint abstraction to a different abstraction for each iterate and the limit.

1.2 Content

Fixpoint transfer theorems state the equality of the abstraction of a least fixpoint and the least fixpoint of an abstract function, under hypotheses such as the commutation of the abstraction function and the iterated function. Sect. **2** presents a new fixpoint transfer theorem that generalizes the well-known theorem on CPOs [6] to the case where, at each iterate, a different concrete function, abstract function, and abstraction function are used. Sect. **3** introduces directed graphs and their classic terminology (finite paths, subpaths, etc.), as well as the totally ordered group of weights. Sect. **4** expresses the (generally infinite) set of (finite) paths of a graph as least fixpoints, using four different possible formulations. Sect. **5** applies the (non extended) fixpoint transfer theorem to these fixpoints, thus exhibiting the common algebraic structure of path problems. Sect. **6** presents an application where the function associating to each pair of vertices the set of paths between them is presented in fixpoint form using a Galois isomorphism. Sect. **7** introduces path weights and a Galois connection between sets of paths and their smallest weight. Sect. **8** applies the (non extended) fixpoint transfer theorem to this Galois connection to find a (greatest) fixpoint characterization of the shortest path between every pair of vertices. However, the function iterated must consider, at each step, every vertex. As each step is performed for every pair of vertices and the number of steps equals the number of vertices, this leads to a $O(n^4)$ cost. Sect. **9** defines elementary (i.e., cycle-free) paths, and Sect. **10** provides four least fixpoint characterizations of them (similar to Sect. **4**). Sect. **11** is the crux of the article. It applies the new fixpoint transfer theorem from Sect. **2** to further simplify the functions iterated to only elementary path. It exploits the fact that each iteration step k uses a slightly different abstraction, that only considers paths using vertices up to vertex k. The commutation condition leads to especially lengthy proofs. The functions iterated in Sect. **11** remain costly as they take care to exactly enumerate elementary paths, pruning any other path. Sect. **12** considers iterating simpler, more efficient functions that do not perform the elementary path check after each concatenation and show that they compute an over-approximation of the set of elementary paths. Sect. **13** presents this fixpoint in a simple algorithmic form by computing iterations through a chaotic iteration scheme. Finally, Sect. **14** applies the path weight abstraction to convert the path enumeration algorithm from Sect. **13** into a shortest-patch algorithm, effectively retrieving exactly the cubic-time Roy-Floy-Warshall algorithm by calculational design. Sect. **15** concludes.

2 Fixpoint abstraction

We write $\mathsf{lfp}^{\sqsubseteq} f$ (respectively $\mathsf{lfp}_a^{\sqsubseteq} f$) for the $\sqsubseteq$-least fixpoint of f (resp. greater than or equal to a), if any. In fixpoint abstraction, it is sometimes necessary to abstract the iterates and their limit differently (similar to the generalization of Scott induction in [5]), as in the following

Theorem 1 (exact abstraction of iterates) *Let* $\langle C, \sqsubseteq, \bot, \bigsqcup\rangle$ *be a cpo,* $\forall i \in \mathbb{N} \,.\, f_i \in C \to C$ *be such that* $\forall x, y \in C \,.\, x \sqsubseteq y \Rightarrow f_i(x) \sqsubseteq f_{i+1}(y)$ *with iterates* $\langle x^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ *defined by* $x^0 = \bot$, $x^{i+1} = f_i(x^i)$, $x^\omega = \bigsqcup_{i\in\mathbb{N}} x^i$. *Then these concrete iterates and* $f \triangleq \dot{\bigsqcup}_{i\in\mathbb{N}} f_i$ *are well-defined.*

Let $\langle \mathcal{A}, \preccurlyeq, 0, \curlyvee\rangle$ *be a cpo,* $\forall i \in \mathbb{N} \,.\, \overline{f}_i \in \mathcal{A} \to \mathcal{A}$ *be such that* $\forall \overline{x}, \overline{y} \in \mathcal{A} \,.\, \overline{x} \preccurlyeq \overline{y} \Rightarrow \overline{f}_i(\overline{x}) \preccurlyeq \overline{f}_{i+1}(\overline{y})$ *with iterates* $\langle \overline{x}^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ *defined by* $\overline{x}^0 = 0$, $\overline{x}^{i+1} = \overline{f}_i(\overline{x}^i)$, $\overline{x}^\omega = \curlyvee_{i\in\mathbb{N}} \overline{x}^i$. *Then these abstract iterates and* $\overline{f} \triangleq \dot{\curlyvee}_{i\in\mathbb{N}} \overline{f}_i$ *are well-defined.*

For all $i \in \mathbb{N} \cup \{\omega\}$, *let* $\alpha_i \in C \to \mathcal{A}$ *be such that* $\alpha_0(\bot) = 0$, $\alpha_{i+1} \circ f_i = \overline{f}_i \circ \alpha_i$, *and* $\alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x_i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x_i)$ *for all increasing chains* $\langle x_i \in C, i \in \mathbb{N}\rangle$. *It follows that* $\alpha_\omega(x^\omega) = \overline{x}^\omega$.

If, moreover, $\forall i \in \mathbb{N} \,.\, f_i \in C \xrightarrow{uc} C$ *is upper-continuous then* $x^\omega = \mathsf{lfp}^{\sqsubseteq} f$. *Similarly* $\overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$ *when the* $\overline{f}_i$ *are upper-continuous. If both the* f_i *and* $\overline{f}_i$ *are upper-continuous then* $\alpha_\omega(\mathsf{lfp}^{\sqsubseteq} f) = \alpha_\omega(x^\omega) = \overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$.

A trivial generalization is to have a different (concrete and) abstract domain at each iteration and the limit (like *e.g.* in *cofibered domains* [20]).

Proof (of Th. 1) $x^0 \triangleq \bot \sqsubseteq x^1$ since $\bot$ is the infimum and if $x^i \sqsubseteq x^{i+1}$ then, by hypothesis, $x^{i+1} \triangleq f_i(x^i) \sqsubseteq f_{i+1}(x^{i+1}) \triangleq x^{i+2}$. Its follows that $\langle x^i, i \in \mathbb{N}\rangle$ is an $\sqsubseteq$-increasing chain so that its lub $x^\omega \triangleq \bigsqcup_{i\in\mathbb{N}} x^i$ is well-defined in the cpo $\langle C, \sqsubseteq\rangle$. The concrete iterates $\langle x^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ are therefore well-defined.

For $x \in C$, reflexivity $x \sqsubseteq x$ implies $f_i(x) \sqsubseteq f_{i+1}(x)$ so $\langle f_i(x), i \in \mathbb{N}\rangle$ is an increasing chain which limit $f(x) \triangleq \bigsqcup_{i\in\mathbb{N}} f_i(x)$ is well-defined in the cpo $\langle C, \sqsubseteq\rangle$.

Similarly, the abstract iterates $\langle \overline{x}^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ and $\overline{f}$ are well-defined.

Let us prove by recurrence on i that $\forall i \in \mathbb{N} \,.\, \alpha_i(x^i) = \overline{x}^i$.

- For the basis, $\alpha_0(x^0) = \alpha_0(\bot) = 0 = \overline{x}^0$.
- Assume, by induction hypothesis, that $\alpha_i(x^i) = \overline{x}^i$. For the induction step,

$$
\begin{aligned}
& \alpha_{i+1}(x^{i+1}) \\
= {} & \alpha_{i+1}(f_i(x^i)) && \text{(def. concrete iterates of the } f_i\text{)} \\
= {} & \overline{f}_i(\alpha_i(x^i)) && \text{(commutation } \alpha_{i+1} \circ f = \overline{f}_i \circ \alpha_i\text{)} \\
= {} & \overline{f}_i(\overline{x}^i) && \text{(ind. hyp.)} \\
= {} & \overline{x}^{i+1} && \text{(def. abstract iterates of the } \overline{f}_i\text{)}
\end{aligned}
$$

It follows that $\alpha_\omega(x^\omega) = \alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x^i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x^i) = \curlyvee_{i\in\mathbb{N}} \overline{x}^i = \overline{x}^\omega$.

If, moreover, $\forall i \in \mathbb{N} \,.\, f_i \in C \xrightarrow{uc} C$ is upper-continuous, then we have

$$
\begin{aligned}
& f(x^\omega) \\
= {} & \bigsqcup_{j\in\mathbb{N}} f_j(\bigsqcup_{i\in\mathbb{N}} x^i) && \text{(def. } f \text{ and } x^\omega\text{)} \\
= {} & \bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} f_j(x^i) && \text{(}\langle x^i, i \in \mathbb{N}\rangle \text{ is an } \sqsubseteq\text{-increasing chain and } f_i \text{ is upper-continuous)} \\
= {} & \bigsqcup_{j\in\mathbb{N}} (\bigsqcup_{i<j} f_j(x^i) \sqcup \bigsqcup_{j=i} f_j(x^i) \sqcup \bigsqcup_{i>j} f_j(x^i)) && \text{(case analysis)}
\end{aligned}
$$

$= \bigsqcup_{j\in\mathbb{N}} (f_j(x^j) \sqcup \bigsqcup_{i>j} f_j(x^i))$

$\wr$since, by recurrence using $x_i \sqsubseteq x_{i+1} \Rightarrow f_i(x_i) \sqsubseteq f_{i+1}(x_{i+1})$, we have $i < j \Rightarrow x_i \sqsubseteq x_j \Rightarrow f_i(x_i) \sqsubseteq f_j(x_j) \Rightarrow \bigsqcup_{i<j} f_j(x^i) \sqsubseteq f_j(x_j)$ and so, by def. lub $\bigsqcup$, $\bigsqcup_{i<j} f_j(x^i) \sqcup \bigsqcup_{j=i} f_j(x^i) = f_j(x^j)$$\wr$

$= (\bigsqcup_{j\in\mathbb{N}} f_j(x^j)) \sqcup (\bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i>j} f_j(x^i))$ $\wr$def. lub $\bigsqcup$$\wr$

$= (\bigsqcup_{j\in\mathbb{N}} f_j(x^j)) \sqcup \bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i\geqslant j} f_i(x^i))$

$\wr$since, $j < i \Rightarrow x_j \sqsubseteq x_i \Rightarrow f_j(x_j) \sqsubseteq f_i(x_i)$ so $(\bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i>j} f_j(x^i)) \sqsubseteq (\bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i>j} f_i(x^i)) = (\bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i\geqslant j} f_i(x^i))$$\wr$

$= (\bigsqcup_{j\in\mathbb{N}} f_j(x^j))$

$\wr(\bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i>j} f_j(x^i)) = (\bigsqcup_{j\in\mathbb{N}} f_j(x^j))$ by $\bigsqcup$ associative, commutative, and idempotent$\wr$

$= \bigsqcup_{i\in\mathbb{N}} x^{i+1} = x^0 \sqcup \bigsqcup_{j\in\mathbb{N}_*} x^j$ $\wr$def. x^{i+1} and $j = i + 1$ is positive$\wr$

$= \bigsqcup_{i\in\mathbb{N}} x^i = x^\omega$ $\wr x^0 = \bot$ is the infimum and def. $x^\omega$$\wr$

Therefore x^ω is a fixpoint of f. Assume that $y \in C$ is a fixpoint of f. Let us prove by recurrence that $\forall i \in \mathbb{N} \,.\, x^i \sqsubseteq y$. For the basis $x^0 = \bot \sqsubseteq y$, by def. of the infimum $\bot$. Assume that $x^i \sqsubseteq y$ by induction hypothesis. Then

x^{i+1}

$= f_i(x^i)$ $\wr$def. abstract iterates$\wr$

$\sqsubseteq f_i(y)$ $\wr$ind. hyp. $x^i \sqsubseteq y$ and f_i upper-continuous hence increasing$\wr$

$\sqsubseteq \bigsqcup_{i\in\mathbb{N}} f_i(y)$ $\wr$def. lub, if it exists$\wr$

$= f(y)$ $\wr\langle f_i(y),\ i \in \mathbb{N}\rangle$ is increasing with well-defined limit $f(y) \triangleq \bigsqcup_{i\in\mathbb{N}} f_i(y)$$\wr$

$= y$ $\wr$fixpoint hypothesis$\wr$

It follows that $x^\omega \triangleq \bigsqcup_{i\in\mathbb{N}} x^i \sqsubseteq y$ proving that $x^\omega = \mathsf{lfp}^{\sqsubseteq} f$ is the $\sqsubseteq$-least fixpoint of f. ◻

Observe that the f_i can be chosen to be all identical equal to $f \in C \xrightarrow{uc} C$ in which case $x \sqsubseteq y \Rightarrow f_i(x) \sqsubseteq f_{i+1}(y)$ follows from f being upper-continuous hence monotonically increasing. Then $\alpha_\omega(\mathsf{lfp}^{\sqsubseteq} f) = \alpha_\omega(x^\omega) = \overline{x}^\omega$. Similarly, the choice $\overline{f}_i = \overline{f} \in \mathcal{A} \xrightarrow{uc} \mathcal{A}$ yields $\alpha_\omega(x^\omega) = \overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$. If, moreover, all α_i are identical, we get the classical [6, theorem 7.1.0.4(3)]

Corollary 1 (exact fixpoint abstraction) *Let* $\langle C, \sqsubseteq, \bot, \bigsqcup\rangle$ *and* $\langle \mathcal{A}, \preccurlyeq, 0, \curlyvee\rangle$ *be cpos,* $f \in C \xrightarrow{uc} C$*,* $\overline{f} \in \mathcal{A} \xrightarrow{uc} \mathcal{A}$*, and* $\alpha \in C \xrightarrow{uc} \mathcal{A}$ *be upper-continuous, such that* $\alpha(\bot) = 0$ *and* $\alpha \circ f = \overline{f} \circ \alpha$*. Then* $\alpha(\mathsf{lfp}^{\sqsubseteq} f) = \mathsf{lfp}^{\preccurlyeq} \overline{f} = \overline{x}^\omega$ *where* $\overline{x}^0 \triangleq 0$*,* $\overline{x}^{i+1} \triangleq \overline{f}_i(\overline{x}^i)$*, and* $\overline{x}^\omega \triangleq \curlyvee_{i\in\mathbb{N}} \overline{x}^i$*.*

By considering $\langle C, \sqsubseteq\rangle = \langle \mathcal{A}, \preccurlyeq\rangle$, $f = \overline{f}$, and the identity abstraction $\alpha(x) = x$, we get Tarski-Kleene-Scott's fixpoint theorem. Th. 1 and Cor. 1 easily extend to fixpoint over-approximation $\alpha(\mathsf{lfp}^{\sqsubseteq} f) \preccurlyeq \mathsf{lfp}^{\preccurlyeq} \overline{f}$.

3 Weighted graphs

3.1 Graphs

A (directed) *graph* or *digraph* $G = \langle V, E\rangle$ is a pair of a set V of *vertices* (or *nodes* or *points*) and a set $E \in \wp(V \times V)$ of *edges* (or *arcs*). A (directed) edge $\langle x, y\rangle \in V$ has *origin* x and *end* y collectively called *extremities* (so the graphs we consider are always directed). A graph is *finite* when the set of V of vertices (hence E) is finite.

A *path* π from y to z in a graph $G = \langle V, E\rangle$ is a finite sequence of vertices $\pi = x_1 \ldots x_n \in V^n$, $n > 1$, starting at *origin* $y = x_1$, finishing at *end* $z = x_n$, and linked by edges $\langle x_i, x_{i+1}\rangle \in E$, $i \in [1, n[$. Let $V^{>1} \triangleq \bigcup_{n>1} V^n$ be the sequences of vertices of length at least 2. Formally the set $\Pi(G) \in \wp(V^{>1})$ of all paths of a graph $G = \langle V, E\rangle$ is

$$\Pi(G) \triangleq \bigcup_{n>1} \Pi^n(G) \tag{1}$$
$$\Pi^n(G) \triangleq \{x_1 \ldots x_n \in V^n \mid \forall i \in [1, n[\;.\; \langle x_i, x_{i+1}\rangle \in E\} \qquad (n > 1)$$

The length $|\pi|$ of the path $\pi = x_1 \ldots x_n \in V^n$ is the number of edges that is $n - 1 > 0$. We do not consider the case $n = 1$ of paths of length 0 with only one vertex since paths must have at least one edge. A *subpath* is a strict contiguous part of another path (without holes and which, being strict, is not equal to that path).

The *vertices of a path* $\pi = x_1 \ldots x_n \in \Pi^n(G)$ of a graph G is the set $\mathbf{V}(\pi) = \{x_1 \ldots x_n\}$ of vertices appearing in that path π.

A *cycle* is a path $x_1 \ldots x_n \in \Pi^n(G)$ with $x_n = x_1$, $n > 1$. Self-loops *i.e.* $\langle x, x\rangle \in E$ yield a cycle $x\,x$ of length 1.

3.2 Totally ordered groups

A *totally (or linearly) ordered group* $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is a group $\langle \mathbb{G}, 0, +\rangle$ with a total order $\leqslant$ on $\mathbb{G}$ satisfying the translation-invariance condition $\forall a, b, c \in \mathbb{G}\;.\;(a \leqslant b) \Rightarrow (a + c \leqslant b + c)$. An element $x \in \mathbb{G}$ of a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is said to be strictly negative if and only if $x \leqslant 0 \wedge x \neq 0$.

If $S \subseteq \mathbb{G}$ then we define $\min S$ to be the greatest lower bound of S in $\mathbb{G}$ or $-\infty$:

$$\begin{aligned}
\min S = m \quad &\Leftrightarrow m \in \mathbb{G} \wedge (\forall x \in S\;.\; m \leqslant x \wedge (\forall y \in S\;.\; y \leqslant x \Rightarrow y \leqslant m) \\
= -\infty &\Leftrightarrow \forall x \in S\;.\;\exists y \in S\;.\; y < x && \text{(where } -\infty \notin \mathbb{G}) \\
= \infty \quad &\Leftrightarrow S = \varnothing && \text{(where } \infty \notin \mathbb{G})
\end{aligned}$$

So if $\mathbb{G}$ has no infimum $\min \mathbb{G} = \max \varnothing = -\infty \notin \mathbb{G}$. Similarly, $\max S$ is the least upper bound of S in $\mathbb{G}$, if any; $-\infty$ otherwise, with $\max \mathbb{G} = \min \varnothing = \infty \notin \mathbb{G}$ when $\mathbb{G}$ has no supremum. Extending $+$ by $x + \infty = \infty + x = \infty + \infty = \infty$ and $x + -\infty = -\infty + x = -\infty + -\infty = -\infty$ for all $x \in \mathbb{G}$, we have $\min\{x + y \mid x \in S_1 \wedge y \in S_2\} = \min S_1 + \min S_2$.

3.3 Weighted graphs

We now equip graphs with weights *e.g.* to measure the distance between vertices. A *weighted graph* on a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is a triple $\langle V, E, \boldsymbol{\omega}\rangle$ of a set V of vertices and a set $E \in \wp(V \times V)$ of edges of a graph $\langle V, E\rangle$, and a weight $\boldsymbol{\omega} \in E \to \mathbb{G}$ mapping edges $\langle x, y\rangle \in E$ to values $\boldsymbol{\omega}(\langle x, y\rangle) \in \mathbb{G}$ taken in the totally ordered group $\mathbb{G}$.

4 Fixpoint characterization of the paths of a graph

The concatenation of sets of finite paths is

$$P \circledcirc Q \triangleq \{x_1 \ldots x_n y_2 \ldots y_m \mid x_1 \ldots x_n \in P \wedge y_1 y_2 \ldots y_m \in Q \wedge x_n = y_1\}. \tag{2}$$

We have the following well-defined fixpoint characterization of the paths of a graph [7, Sect. 4].

Theorem 2 (Fixpoint characterization of the paths of a graph) *The paths of a graph $G = \langle V, E\rangle$ are*

$$\begin{array}{rlll}
\Pi(G) &= \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{L}}_{\Pi}, & \overrightarrow{\mathscr{L}}_{\Pi}(X) \triangleq E \cup X \circledcirc E & \text{(Th.2.a)}\\
&= \mathsf{lfp}^{\subseteq}\, \overleftarrow{\mathscr{R}}_{\Pi}, & \overleftarrow{\mathscr{R}}_{\Pi}(X) \triangleq E \cup E \circledcirc X & \text{(Th.2.b)}\\
&= \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{B}}_{\Pi}, & \overleftrightarrow{\mathscr{B}}_{\Pi}(X) \triangleq E \cup X \circledcirc X & \text{(Th.2.c)}\\
&= \mathsf{lfp}^{\subseteq}_{E}\, \widehat{\mathscr{P}}_{\Pi}, & \widehat{\mathscr{P}}_{\Pi}(X) \triangleq X \cup X \circledcirc X & \text{(Th.2.d)}
\end{array} \qquad \square$$

$\overrightarrow{\mathscr{L}}_{\Pi}$ stands for a forward definition of paths using a left-recursive transformer; $\overleftarrow{\mathscr{R}}_{\Pi}$ stands for a backward definition of paths using a right-recursive transformer; $\overleftrightarrow{\mathscr{B}}_{\Pi}$ stands for a bidirectional definition of paths using a right- and left-recursive transformer; $\widehat{\mathscr{P}}_{\Pi}$ stands for a recursive transformer using paths only which iterations are initialized by edges.

Proof (of Th. 2) We observe that $\bigcup_{i\in\Delta}(X_i \circledcirc E) = \bigcup_{i\in\Delta}\{\pi x y \mid \pi x \in X_i \wedge \langle x, y\rangle \in E\} = \{\pi x y \mid \pi x \in \bigcup_{i\in\Delta} X_i \wedge \langle x, y\rangle \in E\} = (\bigcup_{i\in\Delta} X_i) \circledcirc E$ so that the transformer $\overrightarrow{\mathscr{L}}_{\Pi}$ preserves non-empty joins so is upper-continuous. Same for $\overleftarrow{\mathscr{R}}_{\Pi}$.

Let $\langle X_i, i \in \mathbb{N}\rangle$ be a $\subseteq$-increasing chain of elements of $\wp(V^{>1})$. $\circledcirc$ is componentwise increasing so $\bigcup_{i\in\mathbb{N}}(X_i \circledcirc X_i) \subseteq (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$. Conversely if $\pi \in (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$ then $\pi = \pi_i x \pi_j$ where $\pi_i x \in X_i$ and $x\pi_j \in X_j$. Assume $i \leqslant j$. Because $X_i \subseteq X_j$, $\pi_i x \in X_j$ so $\pi = \pi_i x \pi_j \in X_j \circledcirc X_j \subseteq \bigcup_{k\in\mathbb{N}} X_k \circledcirc X_k$ proving that $\bigcup_{i\in\mathbb{N}}(X_i \circledcirc X_i) \supseteq (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$. We conclude, by antisymmetry, that $\overrightarrow{\mathscr{B}}_{\Pi}$ and $\widehat{\mathscr{P}}_{\Pi}$ are upper-continuous.

It follows, by Tarski-Kleene-Scott's fixpoint theorem, that the least fixpoints do exist.

We consider case (Th.2.c). By upper continuity, we can apply Cor. 1. Let us calculate the iterates $\langle \overrightarrow{\mathscr{B}}_{\Pi}{}^{k}, k \in \mathbb{N}\rangle$ of the fixpoint of transformer $\overrightarrow{\mathscr{B}}_{\Pi}$.

$\overrightarrow{\mathscr{B}}_{\Pi}{}^{0} = \emptyset$, by def. of the iterates.

$\overrightarrow{\mathscr{B}}_{\Pi}{}^{1}(\emptyset) = \overrightarrow{\mathscr{B}}_{\Pi}(\overrightarrow{\mathscr{B}}_{\Pi}{}^{0}) = E = \Pi^2(G)$ contains the paths of length 1 which are made of a single arc. If the graph has no paths longer than mere arcs, all paths are covered after 1 iteration.

Assume, by recurrence hypothesis on k, that $\overrightarrow{\mathscr{B}}_{\Pi}{}^{k} = \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$ contains exactly all paths of G of length less than or equal to 2^{k-1}. We have

$$\begin{aligned}
\overrightarrow{\mathscr{B}}_{\Pi}{}^{k+1} &\triangleq \overrightarrow{\mathscr{B}}_{\Pi}(\overrightarrow{\mathscr{B}}_{\Pi}{}^{k}) && \text{(def. iterates)}\\
&= E \cup \overrightarrow{\mathscr{B}}_{\Pi}{}^{k} \circledcirc \overrightarrow{\mathscr{B}}_{\Pi}{}^{k} && \text{(def. } \overrightarrow{\mathscr{B}}_{\Pi}\text{)}\\
&= E \cup \{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \overrightarrow{\mathscr{B}}_{\Pi}{}^{k} \wedge x_n y_2 \dots y_m \in \overrightarrow{\mathscr{B}}_{\Pi}{}^{k}\} && \text{(def. } \circledcirc\text{)}\\
&= E \cup \{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \bigcup_{n=2}^{2^{k-1}} \Pi^n(G) \wedge x_n y_2 \dots y_m \in \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)\}\\
& && \text{(ind. hyp.)}\\
&= E \cup \bigcup_{n=3}^{2^k} \Pi^n(G)
\end{aligned}$$

((⊆) the concatenation of two paths of length at least 1 and at most 2^{k-1} is at least of length 2 and at most of length $2 \times 2^{k-1} = 2^k$.
(⊇) Conversely, any path of length at most 2^k has either length 1 in E or can be decomposed into two paths $\pi = x_1 \dots x_n$ and $\pi' = x_n y_2 \dots y_m$ of length at most 2^{k-1}. By induction hypothesis, $\pi, \pi' \in \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$)

By recurrence on k, for all $k \in \mathbb{N}_*$, $\overrightarrow{\mathscr{B}}_{\Pi}{}^{k} = \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$ contains exactly all paths from x to y of length less than or equal to 2^{k-1}.

Finally, we must prove that the limit $\mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{B}}_{\Pi} = \bigcup_{k\in\mathbb{N}} \overrightarrow{\mathscr{B}}_{\Pi}{}^{k}$ is $\Pi(G)$ that is contains exactly all paths of G.

Any path in $\Pi(G)$ has a length $n > 0$ such that $n \leqslant 2^{k-1}$ for some $k > 0$ so belongs to $\overrightarrow{\mathscr{B}}_{\Pi}{}^{n}(\emptyset)$ hence to the limit, proving $\Pi(G) \subseteq \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{B}}_{\Pi}$.

Conversely any path in $\mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{B}}_{\Pi} = \bigcup_{k\in\mathbb{N}} \overrightarrow{\mathscr{B}}_{\Pi}{}^{k}$ belongs to some iterate $\overrightarrow{\mathscr{B}}_{\Pi}{}^{k}$ which contains exactly all paths of length less than or equal to 2^k so belongs to $\Pi^{2^k}(G)$ hence to $\Pi(G)$, proving $\mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{B}}_{\Pi} \subseteq \Pi(G)$. By antisymmetry $\Pi(G) = \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{B}}_{\Pi}$.

The equivalent form $\widehat{\mathscr{P}}_{\Pi}$ follows from $\mathsf{lfp}^{\sqsubseteq} f = \mathsf{lfp}^{\sqsubseteq}\, \lambda x \bullet x \sqcup f(x)$ and $\mathsf{lfp}^{\sqsubseteq}\, \lambda x \bullet a \sqcup f(x) = \mathsf{lfp}_a^{\sqsubseteq} f$ when $a \sqsubseteq f(a)$. The proofs for (Th.2.a,b) are similar with the k^{th}-iterate of the form $\bigcup_{n=2}^{k} \Pi^n(G)$. □

5 Abstraction of the paths of a graph

A *path problem* in a graph $G = \langle V,\ E\rangle$ consists in specifying/computing an abstraction $\alpha(\Pi(G))$ of its paths $\Pi(G)$ defined by a Galois connection

$$\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xrightleftharpoons[\alpha]{\gamma} \langle A,\ \sqsubseteq,\ \sqcup\rangle.$$

A path problem can be solved by a fixpoint definition/computation.

Theorem 3 (Fixpoint characterization of a path problem) *Let* $G = \langle V,\ E\rangle$ *be a graph with paths* $\Pi(G)$ *and* $\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xrightleftharpoons[\alpha]{\gamma} \langle A,\ \sqsubseteq,\ \sqcup\rangle$.

$$\begin{aligned}
\alpha(\Pi(G)) &= \mathsf{lfp}^{\sqsubseteq}\, \overrightarrow{\mathscr{L}}^{\sharp}_{\Pi}, & \overrightarrow{\mathscr{L}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} \alpha(E) & \text{(Th.3.a)}\\
&= \mathsf{lfp}^{\sqsubseteq}\, \overleftarrow{\mathscr{R}}^{\sharp}_{\Pi}, & \overleftarrow{\mathscr{R}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup \alpha(E) \mathbin{\overline{\circledcirc}} X & \text{(Th.3.b)}\\
&= \mathsf{lfp}^{\sqsubseteq}\, \overleftrightarrow{\mathscr{B}}^{\sharp}_{\Pi}, & \overleftrightarrow{\mathscr{B}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} X & \text{(Th.3.c)}\\
&= \mathsf{lfp}^{\sqsubseteq}_{\alpha(E)}\, \widehat{\mathscr{P}}^{\sharp}_{\Pi}, & \widehat{\mathscr{P}}^{\sharp}_{\Pi}(X) &\triangleq X \sqcup X \mathbin{\overline{\circledcirc}} X & \text{(Th.3.d)}
\end{aligned}$$

where $\alpha(X) \mathbin{\overline{\circledcirc}} \alpha(Y) = \alpha(X \circledcirc Y)$. □

Proof (of Th. 3) All cases are similar. Let us check the commutation for (Th.3.c).

$$\begin{aligned}
&\alpha(\overleftrightarrow{\mathscr{B}}_{\Pi}(X))\\
=\ &\alpha(E \cup X \circledcirc X) && \text{≀def. (Th.2.c) of } \overleftrightarrow{\mathscr{B}}_{\Pi}(X)\text{≀}\\
=\ &\alpha(E) \sqcup \alpha(X \circledcirc X)\\
& && \text{≀the abstraction of Galois connections preserves existing joins≀}\\
=\ &\alpha(E) \sqcup \alpha(X) \mathbin{\overline{\circledcirc}} \alpha(X) && \text{≀by hyp.≀}\\
=\ &\overleftrightarrow{\mathscr{B}}^{\sharp}_{\Pi}(\alpha(X)) && \text{≀def. (Th.3.c) of } \overleftrightarrow{\mathscr{B}}^{\sharp}_{\Pi}\text{≀}
\end{aligned}$$

We conclude by Th. 2 and exact least fixpoint abstraction Cor. 1. The equivalent form $\widehat{\mathscr{P}}^{\sharp}_{\Pi}$ follows from $\mathsf{lfp}^{\sqsubseteq} f = \mathsf{lfp}^{\sqsubseteq}\, \lambda\, x \bullet x \sqcup f(x)$ and $\mathsf{lfp}^{\sqsubseteq}\, \lambda\, x \bullet a \sqcup f(x) = \mathsf{lfp}^{\sqsubseteq}_{a} f$ when $a \sqsubseteq f(a)$. □

An essential remark is that the fixpoint definitions of the set of paths in $\wp(V^{>1})$ of a graph $G = \langle V,\ E\rangle$ in Th. 2 based on the primitives E, $\cup$, and $\circledcirc$ are preserved in Th. 3 by the abstraction $\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xrightleftharpoons[\alpha]{\gamma} \langle A,\ \sqsubseteq,\ \sqcup\rangle$ for the primitives $\alpha(E)$, $\sqcup$, and $\overline{\circledcirc}$ on A, which explains the origin of the observation by [2,14,15,9,11] that path problems all have the same algebraic structure.

6 Calculational design of the paths between any two vertices

As a direct application of Th. 3, let us consider the abstraction of all paths $\Pi(G)$ into the paths between any two vertices. This is $\mathsf{p} \triangleq \alpha^{\circ\circ}(\Pi(G))$ with the projection abstraction

$$\alpha^{\circ\!\!-\!\!\circ}(X) \triangleq \lambda\,(y,z) \bullet \{x_1 \ldots x_n \in X \mid y = x_1 \wedge x_n = z\}$$
$$\gamma^{\circ\!\!-\!\!\circ}(\mathsf{p}) \triangleq \bigcup_{\langle x,\, y\rangle \in V\times V} \mathsf{p}(x,y)$$

such that

$$\langle \wp(V^{>1}),\ \subseteq,\ \cup\rangle \xleftarrow[\alpha^{\circ\!\!-\!\!\circ}]{\gamma^{\circ\!\!-\!\!\circ}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq},\ \dot{\cup}\rangle \tag{3}$$

where $\mathsf{p} \mathrel{\dot{\subseteq}} \mathsf{p}' \Leftrightarrow \forall x, y \in V\ .\ \mathsf{p}(x,y) \subseteq \mathsf{p}'(x,y)$ and $(\dot{\bigcup\limits_{i\in\Delta}}\mathsf{p}_i)(x,y) \triangleq \bigcup\limits_{i\in\Delta}(\mathsf{p}_i(x,y))$ are defined pointwise.

By (1) and the abstraction in Galois connections preserves existing joins, we have

$$\mathsf{p}(y,z) \triangleq \bigcup_{n\in\mathbb{N}_*} \mathsf{p}^n(y,z) \tag{4}$$
$$\begin{aligned}\mathsf{p}^n(y,z) &\triangleq \{x_1 \ldots x_n \in \Pi^n(G) \mid y = x_1 \wedge x_n = z\}\\ &= \{x_1 \ldots x_n \in V^n \mid y = x_1 \wedge x_n = z \wedge \forall i \in [1,n[\ .\ \langle x_i,\ x_{i+1}\rangle \in E\}.\end{aligned}$$

$\mathsf{p}(x,x)$ is empty if and only if there is no cycle from x to x (which requires, in particular, that the graph has no self-loops *i.e.* $\forall x \in V\ .\ \langle x,\ x\rangle \notin E$). We define the concatenation of finite paths

$$\begin{aligned} x_1 \ldots x_n \odot y_1 y_2 \ldots y_m &\triangleq x_1 \ldots x_n y_2 \ldots y_m && \text{if } x_n = y_1\\ &\triangleq \mathsf{undefined} && \text{otherwise}\end{aligned} \tag{5}$$

As a direct application of the path problem Th. 3, we have the following fixpoint characterization of the paths of a graph between any two vertices [7, Sect. 5], which, by Kleene-Scott fixpoint theorem, yields an iterative algorithm (converging in finitely many iterations for graphs without infinite paths).

Theorem 4 (Fixpoint characterization of the paths of a graph between any two vertices) *Let* $G = \langle V,\ E\rangle$ *be a graph. The paths between any two vertices of* G *are* $\mathsf{p} = \alpha^{\circ\!\!-\!\!\circ}(\Pi(G))$ *such that*

$$\begin{aligned}
\mathsf{p} &= \mathsf{lfp}^{\dot{\subseteq}}\ \overrightarrow{\mathscr{L}}^{\circ\!\!-\!\!\circ}_{\Pi}, & \overrightarrow{\mathscr{L}}^{\circ\!\!-\!\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathrel{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}} \dot{E} && \text{(Th.4.a)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}\ \overleftarrow{\mathscr{R}}^{\circ\!\!-\!\!\circ}_{\Pi}, & \overleftarrow{\mathscr{R}}^{\circ\!\!-\!\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathrel{\dot{\cup}} \dot{E} \mathbin{\dot{\circledcirc}} \mathsf{p} && \text{(Th.4.b)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}\ \overleftrightarrow{\mathscr{B}}^{\circ\!\!-\!\!\circ}_{\Pi}, & \overleftrightarrow{\mathscr{B}}^{\circ\!\!-\!\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathrel{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}} \mathsf{p} && \text{(Th.4.c)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\ \widehat{\mathscr{P}}^{\circ\!\!-\!\!\circ}_{\Pi}, & \widehat{\mathscr{P}}^{\circ\!\!-\!\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \mathsf{p} \mathrel{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}} \mathsf{p} && \text{(Th.4.d)}
\end{aligned}$$

where $\dot{E} \triangleq \lambda\, x, y \bullet (E \cap \{\langle x,\ y\rangle\})$ *and* $\mathsf{p}_1 \mathbin{\dot{\circledcirc}} \mathsf{p}_2 \triangleq \lambda\, x, y \bullet \bigcup\limits_{z\in V} \mathsf{p}_1(x,z) \circledcirc \mathsf{p}_2(z,y)$.

□

Proof (of Th. 4) We apply Th. 3 with $\alpha^{\circ\!\!-\!\!\circ}(E) = \lambda\, x, y \bullet (E \cap \{\langle x,\ y\rangle\}) = \dot{E}$ and

$$\begin{aligned}&\alpha^{\circ\!\!-\!\!\circ}(X \circledcirc Y)\\ =\ &\lambda\,(x,y) \bullet \{z_1 \ldots z_n \in X \circledcirc Y \mid x = z_1 \wedge z_n = y\} && \text{(def. (3) of } \alpha^{\circ\!\!-\!\!\circ}\text{)}\end{aligned}$$

$= \lambda\,(x, y) \bullet \{z_1 \ldots z_n \in \{x_1 \ldots x_k y_2 \ldots y_m \mid x_1 \ldots x_k \in X \wedge x_k y_2 \ldots y_m \in Y\} \mid x = z_1 \wedge z_n = y\}$

$\wr$def. (2) of $\circledcirc$$\int$

$= \lambda\,(x, y) \bullet \bigcup_{z \in V} \{x x_2 \ldots x_{k-1} z y_2 \ldots , y_{m-1} y \mid x x_2 \ldots x_{k-1} z \in X \wedge z y_2 \ldots y_{m-1} y \in Y\}$

$\wr$def. $\in$ and $\cup$ with $x = x_1,\ y_m = y$, and $z = x_k$$\int$

$= \lambda\,(x, y) \bullet \bigcup_{z \in V} \{x x_2 \ldots x_{k-1} z \odot z y_2 \ldots y_{m-1} y \mid x x_2 \ldots x_{k-1} z \in X \wedge z y_2 \ldots y_{m-1} y \in Y\}$

$\wr$def. (5) of $\odot$$\int$

$= \lambda\,(x, y) \bullet \bigcup_{z \in V} \{p \odot p' \mid p \in \alpha^{\circ\!\!-\!\!\circ}(X)(y, z) \wedge p' \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\}$

$\wr$def. $\alpha^{\circ\!\!-\!\!\circ}(X)$ with $p = x x_2 \ldots x_{k-1} z$ and $p' = z y_2 \ldots y_{m-1} y$$\int$

$= \alpha^{\circ\!\!-\!\!\circ}(X) \mathbin{\dot{\mathring{\circledcirc}}} \alpha^{\circ\!\!-\!\!\circ}(Y)$

by defining $X \mathbin{\dot{\mathring{\circledcirc}}} Y \triangleq \lambda\,(x, y) \bullet \bigcup_{z \in V} \{p \odot p' \mid p \in X(y, z) \wedge p' \in Y(z, y)\} = \lambda\,(x, y) \bullet \bigcup_{z \in V} X(y, z) \circledcirc Y(z, y)$ by (2) and (5). □

7 Shortest distances between any two vertices of a weighted graph

We now consider weighted graphs $\langle V,\ E,\ \boldsymbol{\omega}\rangle$ on a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ and extend weights from edges to paths. The weight of a path is

$$\boldsymbol{\omega}(x_1 \ldots x_n) \triangleq \sum_{i=1}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle) \tag{6}$$

which is 0 when $n \leqslant 1$ and $\sum_{i=1}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle)$ when $n > 1$, in particular $\boldsymbol{\omega}(\langle x_1,\ x_2\rangle)$ when $n = 2$. The (minimal) weight of a set of paths is

$$\boldsymbol{\omega}(P) \triangleq \min\{\boldsymbol{\omega}(\pi) \mid \pi \in P\}. \tag{7}$$

We have $\boldsymbol{\omega}(\bigcup_{i \in \Delta} P_i) = \min_{i \in \Delta} \boldsymbol{\omega}(P_i)$ so a Galois connection

$$\langle \wp(\bigcup_{n \in \mathbb{N}_*} V^n),\ \subseteq\rangle \xleftrightarrow[\boldsymbol{\omega}]{\gamma_{\boldsymbol{\omega}}} \langle \mathbb{G} \cup \{-\infty, \infty\},\ \geqslant\rangle$$

between path sets and the complete lattice $\langle \mathbb{G} \cup \{-\infty, \infty\}, \geqslant, \infty, -\infty, \min, \max\rangle$ and $\gamma_{\boldsymbol{\omega}}(\mathsf{d}) \triangleq \{\pi \in \bigcup_{n \in \mathbb{N}_*} V^n \mid \boldsymbol{\omega}(\pi) \geqslant \mathsf{d}\}$.

Extending pointwise to $V \times V \to \wp(\bigcup_{n \in \mathbb{N}_*} V^n)$ with $\dot{\boldsymbol{\omega}}(\mathsf{p})\langle x,\ y\rangle \triangleq \boldsymbol{\omega}(\mathsf{p}(x, y))$, $\mathsf{d} \mathrel{\dot{\leqslant}} \mathsf{d}' \triangleq \forall x, y \mathrel{.} \mathsf{d}\langle x,\ y\rangle \leqslant \mathsf{d}'\langle x,\ y\rangle$, and $\dot{\geqslant}$ is the inverse of $\dot{\leqslant}$, we have

$$\langle V \times V \to \wp(\bigcup_{n \in \mathbb{N}_*} V^n),\ \dot{\subseteq}\rangle \xleftrightarrow[\dot{\boldsymbol{\omega}}]{\dot{\gamma}_{\boldsymbol{\omega}}} \langle V \times V \to \mathbb{G} \cup \{-\infty, \infty\},\ \dot{\geqslant}\rangle. \tag{8}$$

The distance $\mathsf{d}(x, y)$ between an origin $x \in V$ and an extremity $y \in V$ of a weighted finite graph $G = \langle V, E, \boldsymbol{\omega}\rangle$ on a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is the length $\boldsymbol{\omega}(\mathsf{p}(x, y))$ of the shortest path between these vertices

$$\mathsf{d} \triangleq \dot{\boldsymbol{\omega}}(\mathsf{p})$$

where p has a fixpoint characterization given by Th. 4.

8 Calculational design of the shortest distances between any two vertices

The shortest distance between vertices of a weighted graph is a path problem solved by Th. 3, the composition of the abstractions and (8) and (3), and the path abstraction Th. 3. Th. 5 is based on (Th.3.d), (Th.3.a—c) provide three other solutions.

Theorem 5 ***(Fixpoint characterization of the shortest distances of a graph)*** *Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a graph weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$. Then the distances between any two vertices are*

$$\mathsf{d} = \dot{\boldsymbol{\omega}}(\mathsf{p}) = \mathsf{gfp}^{\dot{\leqslant}}_{E^{\boldsymbol{\omega}}}\ \widehat{\mathscr{P}}^{\delta}_{G} \qquad \textit{where} \tag{Th.5}$$

$$E^{\boldsymbol{\omega}} \triangleq \lambda\,(x, y) \bullet [\![\, \langle x,\ y\rangle \in E \;?\; \boldsymbol{\omega}(x, y) \;\text{\textdegree}\; \infty \,]\!]$$

$$\widehat{\mathscr{P}}^{\delta}_{G}(X) \triangleq \lambda\,(x, y) \bullet\ \min\{X(x, y), \min_{z\in V}\{X(x, z) + X(z, y)\}\} \qquad \square$$

Proof (of Th. 5) We apply Th. 3 with abstraction $\dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}$ so that we have to abstract the transformers in Th. 4 using an exact fixpoint abstraction of Cor. 1. The initialization and commutation condition yield the transformers by calculational design.

$$
\begin{aligned}
&- \quad \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(E)(x, y)\\
&= \boldsymbol{\omega}(E \cap \{\langle x,\ y\rangle\}) && \text{(as proved for Th. 4 and def. } \dot{\boldsymbol{\omega}}\text{)}\\
&= [\![\, \langle x,\ y\rangle \in E \;?\; \boldsymbol{\omega}(x, y) \;\text{\textdegree}\; \min\varnothing \,]\!] && \text{(def. } \cap\text{, conditional, and } \boldsymbol{\omega}\text{)}\\
&= [\![\, \langle x,\ y\rangle \in E \;?\; \boldsymbol{\omega}(x, y) \;\text{\textdegree}\; \infty \,]\!] && \text{(def. min)}
\end{aligned}
$$

$$
\begin{aligned}
&- \quad \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(X \circledcirc Y)(x, y)\\
&= \dot{\boldsymbol{\omega}}(\alpha^{\circ\!\!-\!\!\circ}(X)\ \ddot{\circledcirc}\ \alpha^{\circ\!\!-\!\!\circ}(Y))(x, y) && \text{(as proved for Th. 4)}\\
&= \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(X)\ \ddot{\circledcirc}\ \alpha^{\circ\!\!-\!\!\circ}(Y))(x, y)) && \text{(pointwise def. (8) of } \dot{\boldsymbol{\omega}}\text{)}\\
&= \boldsymbol{\omega}(\bigcup_{z\in V} \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \circledcirc \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y))) && \text{(def. } \ddot{\circledcirc} \text{ in Th. 4)}\\
&= \min_{z\in V} \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \circledcirc \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y))) && \text{(Galois connection (7))}\\
&= \min_{z\in V} \boldsymbol{\omega}(\{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})\\
& && \text{(def. (2) of } \circledcirc\text{)}\\
&= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})
\end{aligned}
$$

$$\text{\wr def. (7) of } \boldsymbol{\omega} \int$$

$$= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n) + \boldsymbol{\omega}(x_n y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!-\!\circ}(X)(x,z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!-\!\circ}(Y)(z,y)\})$$

$$\text{\wr def. (6) of } \boldsymbol{\omega} \int$$

$$= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n) + \boldsymbol{\omega}(y_1 y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!-\!\circ}(X)(x,z) \wedge y_1 y_2 \dots y_m \in \alpha^{\circ\!-\!\circ}(Y)(z,y)\})$$

$$\text{\wr def. } \alpha^{\circ\!-\!\circ} \text{ so that } x_1 = x,\ x_n = y_1 = z, \text{ and } y_m = y \int$$

$$= \min_{z\in V} \min\{\boldsymbol{\omega}(x_1 \dots x_n) \mid x_1 \dots x_n \in \alpha^{\circ\!-\!\circ}(X)(x,z)\} + \min\{\boldsymbol{\omega}(y_1 y_2 \dots y_m) \mid y_1 y_2 \dots y_m \in \alpha^{\circ\!-\!\circ}(Y)(z,y)\} \qquad \text{\wr min of a sum} \int$$

$$= \min_{z\in V} \min\{\boldsymbol{\omega}(\pi_1) \mid \pi_1 \in \alpha^{\circ\!-\!\circ}(X)(x,z)\} + \min\{\boldsymbol{\omega}(\pi_2) \mid \pi_2 \in \alpha^{\circ\!-\!\circ}(Y)(z,y)\}$$

$$\text{\wr letting } \pi_1 = x_1 \dots x_n \text{ and } \pi_2 = y_1 y_2 \dots y_m \int$$

$$= \min_{z\in V} \boldsymbol{\omega}(\alpha^{\circ\!-\!\circ}(X)(x,z)) + \boldsymbol{\omega}(\alpha^{\circ\!-\!\circ}(Y)(z,y)) \qquad \text{\wr def. (7) of } \boldsymbol{\omega} \int$$

$$= \min_{z\in V} \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!-\!\circ}(X)(x,z) + \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!-\!\circ}(Y)(z,y) \qquad \text{\wr pointwise def. (8) of } \dot{\boldsymbol{\omega}} \int$$

By Th. 3 and (Th.4.d), we get the transformer $\widehat{\mathscr{P}}^{\delta}_{G}$. □

Of course the greatest fixpoint in Th. 5 is not computable for infinite graphs. For finite graphs, there is a problem with cycles with strictly negative weights. As shown by the graph $\langle\{x\}, \{\langle x, x\rangle, \boldsymbol{\omega}\rangle$ with $\boldsymbol{\omega}(\langle x, x\rangle) = -1$, the minimal distance between the extremities x and x of the paths $\{x^n \mid n > 1\}$ is $-\infty$. It is obtained as the limit of an infinite iteration for the greatest fixpoint in Th. 5. Following Roy-Floyd-Warshall, we will assume that the graph has no cycle with negative weight in which case the iterative computation of the greatest fixpoint in Th. 5 does converge in finite time to the shortest distance between any two vertices.

For a finite graph of n vertices, the computation of $\mathsf{gfp}^{\leqslant}_{E^{\boldsymbol{\omega}}} \widehat{\mathscr{P}}^{\delta}_{G}$ in (Th.5) has to consider all pairs of vertices in n^2, for each such pair $\langle x, y\rangle$ the n vertices $z \in V$, and n iterations may be necessary to converge along an elementary path (with no cycles) going through all vertices, so considering elementary paths only, the computation would be in $\mathcal{O}(n^4)$.

However, the iteration in Roy-Floyd-Warshall algorithm is much more efficient in $\mathcal{O}(n^3)$, since it does not consider all elementary paths in the graph but only simple paths that over-approximate elementary paths, which simplifies the iterated function (from linear to constant time for each pair of vertices). Let us design the Roy-Floyd-Warshall algorithm by calculus.

9 Elementary paths and cycles

A cycle is *elementary* if and only if it contains no internal subcycle (*i.e.* subpath which is a cycle). A path is *elementary* if and only if it contains no subpath which is an internal cycle (so an elementary cycle is an elementary path). The only vertices that can occur twice in an elementary path are its extremities in which case it is an elementary cycle.

Lemma 1 (elementary path) *A path* $x_1 \ldots x_n \in \Pi^n(G)$ *is elementary if and only if*

$$\mathsf{elem?}(x_1 \ldots x_n) \triangleq (\forall i, j \in [1,n] \,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \vee \qquad \text{(Lem.1)}$$
$$(x_1 = x_n \wedge \forall i, j \in [1,n[\,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \quad \textit{(case of a cycle)}$$

is true.

Proof (of Lem. 1)
— The necessary condition (*i.e.* $x_1 \ldots x_n \in \Pi^n(G)$ is elementary implies that $\mathsf{elem?}(x_1 \ldots x_n)$) is proved contrapositively.

$$\begin{aligned}
&\neg(\mathsf{elem?}(x_1 \ldots x_n)) \\
=\ &\neg((\forall i, j \in [1,n] \,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \vee (x_1 = x_n \wedge \mathsf{elem?}(x_1 \ldots x_n))) \quad \text{\{def. elem?\}} \\
=\ &(\exists i, j \in [1,n] \,.\, i \neq j \wedge x_i = x_j) \wedge ((x_1 = x_n) \Rightarrow (\exists i, j \in [1,n[\,.\, i \neq j \wedge x_i \neq x_j)) \\
&\hfill \text{\{De Morgan laws\}}
\end{aligned}$$

By $\exists i, j \in [1,n] \,.\, i \neq j \wedge x_i = x_j$ the path $x_1 \ldots x_n$ must have a cycle, but this is not forbidden if $x_1 = x_n$. In that case, the second condition $(x_1 = x_n) \Rightarrow (\exists i, j \in [1,n[\,.\, i \neq j \wedge x_i \neq x_j)$ implies that there is a subcycle within $x_1 \ldots x_{n-1}$, so the cycle $x_1 \ldots x_{n-1} x_1$ is not elementary.

— Conversely, the sufficient condition ($\mathsf{elem?}(x_1 \ldots x_n) \Rightarrow x_1 \ldots x_n \in \Pi^n(G)$ is elementary) is proved by *reductio ad absurdum*. Assume $\mathsf{elem?}(x_1 \ldots x_n)$ and $x_1 \ldots x_n \in \Pi^n(G)$ is not elementary so has an internal subcycle.

- If $x_1 = x_n$, the internal subcycle is $x_1 \ldots x_{n-1} = \pi_1 a \pi_2 a \pi_3$ so $\exists i, j \in [1,n[\,.\, i \neq j \wedge x_i \neq x_j$ in contradiction with $\mathsf{elem?}(x_1 \ldots x_n)$.
- Otherwise $x_1 \neq x_n$ and the internal subcycle has the form $x_1 \ldots x_n = \pi_1 a \pi_2 a \pi_3$ where, possibly $\pi_1 a = x_1$ or $a \pi_3 = x_n$, but not both, so $\exists i, j \in [1,n] \,.\, i \neq j \wedge x_i \neq x_j$ in contradiction with $\mathsf{elem?}(x_1 \ldots x_n)$.

□

10 Calculational design of the elementary paths between any two vertices

Restricting paths to elementary ones is the abstraction

$$\alpha^{\ni}(P) \triangleq \{\pi \in P \mid \mathsf{elem?}(\pi)\}$$
$$\gamma^{\ni}(\overline{P}) \triangleq \overline{P} \cup \{\pi \in \wp(V^{>1}) \mid \neg\mathsf{elem?}(\pi)\}$$

Notice that, by (Lem.1), cycles (such as x, x for a self-loop $\langle x,\ x\rangle \in E$) are not excluded, provided it is through the path extremities. This exclusion abstraction is a Galois connection.

$$\langle \wp(V^{>1}),\ \subseteq\rangle \xrightleftharpoons[\alpha^{\ni}]{\gamma^{\ni}} \langle \wp(V^{>1}),\ \subseteq\rangle$$

which extends pointwise between any two vertices

$$\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xleftarrow[\dot{\alpha}^{\ni}]{\dot{\gamma}^{\ni}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle$$

The following Lem. 2 provides a necessary and sufficient condition for the concatenation of two elementary paths to be elementary.

Lemma 2 (concatenation of elementary paths) *If $x\pi_1 z$ and $z\pi_2 y$ are elementary paths then their concatenation $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y$ is elementary if and only if*

$$\begin{aligned}\textsf{elem-conc?}(x\pi_1 z, z\pi_2 y) \triangleq{} & (x \neq z \wedge y \neq z \wedge \mathbf{V}(x\pi_1 z) \cap \mathbf{V}(\pi_2 y) = \varnothing) \\ & \vee (x = y \neq z \wedge \mathbf{V}(\pi_1 z) \cap \mathbf{V}(\pi_2) = \varnothing)\end{aligned} \quad \text{(Lem.2)}$$

is true.

Proof (of Lem. 2) Assuming $x\pi_1 z$ and $z\pi_2 y$ to be elementary, we must prove that $\textsf{elem-conc?}(x\pi_1 z, z\pi_2 y)$ is true $\Leftrightarrow$ $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y$ is elementary.

— We prove the necessary condition ($\pi_1 \odot \pi_2$ is elementary $\Rightarrow \textsf{elem-conc?}(\pi_1, \pi_2)$) by contraposition ($\neg\textsf{elem-conc?}(\pi_1, \pi_2) \Rightarrow \pi_1 \odot \pi_2$ has an internal cycle). We have

$$\begin{aligned} & \neg((x \neq z \wedge y \neq z \wedge \mathbf{V}(x\pi_1 z) \cap \mathbf{V}(\pi_2 y) = \varnothing) \vee (x = y \wedge x \neq z \wedge y \neq z \wedge \mathbf{V}(\pi_1 z) \cap \\ & \mathbf{V}(\pi_2) = \varnothing)) \\ = {} & (x = z \vee y = z \vee (\mathbf{V}(x\pi_1 z) \cap \mathbf{V}(\pi_2 y) \neq \varnothing) \wedge (x \neq y \vee x = z \vee y = z \vee \mathbf{V}(\pi_1 z) \cap \mathbf{V}(\pi_2) \neq \\ & \varnothing)) \qquad \text{(de Morgan laws)}\end{aligned}$$

- If $x = z$ then $x\pi_1 x\pi_2 y$ has a cycle and is not elementary;
- else, if $y = z$ then $x\pi_1 y\pi_2 y$ has a cycle and is not elementary;
- else $x \neq z \wedge y \neq z$, and then
  - either $x \neq z \wedge y \neq z \wedge x = y$ so $\mathbf{V}(\pi_1 z) \cap \mathbf{V}(\pi_2) \neq \varnothing$. There are two cases
    * either $\mathbf{V}(\pi_1) \cap \mathbf{V}(\pi_2) \neq \varnothing$ so $\pi_1 = \pi_1' a \pi_1''$ and $\pi_2 = \pi_2' a \pi_2''$ and therefore $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1' a \pi_1'' z \pi_2' a \pi_2'' x$ has an internal cycle $a\pi_1'' z\pi_2' a$,
    * or $z \in \mathbf{V}(\pi_2)$ so $\pi_2 = \pi_2' z \pi_2''$ and therefore $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' z\pi_2'' x$ has an internal cycle $z\pi_2' z$;
  - otherwise $x \neq z \wedge y \neq z \wedge x \neq y$ and we have $\mathbf{V}(x\pi_1 z) \cap \mathbf{V}(\pi_2 y) \neq \varnothing$. By cases.
    * If x appears in $\pi_2 y$ that is in π_2 since $x \neq y$ we have $\pi_2 = \pi_2' x \pi_2''$ and then $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' x\pi_2'' y$ has an internal cycle $x\pi_1 z\pi_2' x$;
    * Else, if $\mathbf{V}(\pi_1) \cap \mathbf{V}(\pi_2 y) \neq \varnothing$ then
      · Either $\mathbf{V}(\pi_1) \cap \mathbf{V}(\pi_2) \neq \varnothing$ so $\pi_1 = \pi_1' a \pi_1''$ and $\pi_2 = \pi_2' a \pi_2''$ and therefore $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y = x\pi_1' a\pi_1'' z\pi_2' a\pi_2'' x$ has an internal cycle $a\pi_1'' z\pi_2' a$,
      · Or $y \in \mathbf{V}(\pi_1)$ so $\pi_1 = \pi_1' y \pi_1''$ and then $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y = x\pi_1' y\pi_1'' z\pi_2 y$ has an internal cycle $y\pi_1'' z\pi_2 y$;
    * Otherwise, $z \in \mathbf{V}(\pi_2 y) \neq \varnothing$ and then
      · Either $z \in \mathbf{V}(\pi_2)$ so $\pi_2 = \pi_2' z \pi_2''$ and $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' z\pi_2'' y$ has an internal cycle $z\pi_2' z$,
      · Or $z = y$ and $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2 z$ has an internal cycle $z\pi_2 z$.

— We prove that the condition is sufficient ($\textsf{elem-conc?}(\pi_1, \pi_2) \Rightarrow \pi_1 \odot \pi_2$ is elementary) by *reductio ad absurdum.* Assume $x\pi_1 z$, and $z\pi_2 y$ are elementary, $\textsf{elem-conc?}(x\pi_1 z, z\pi_2 y)$ holds, but that $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y$ is not elementary.

- if the internal cycle is in $x\pi_1 z$ then, by hypothesis, $x = z$ so $\textsf{elem-conc?}(x\pi_1 z, z\pi_2 y)$ does not hold, a contradiction;
- else, if the internal cycle is in $z\pi_2 y$ then, by hypothesis, $z = y$ so $\textsf{elem-conc?}(x\pi_1 z, z\pi_2 y)$ does not hold, a contradiction;
- otherwise, the internal cycle is neither in $x\pi_1 z$ nor in $z\pi_2 y$ so $\mathbf{V}(x\pi_1 z) \cap \mathbf{V}(\pi_2 y) \neq \varnothing$. Since $\textsf{elem-conc?}(\pi_1, \pi_2)$ holds, it follows that $x = y \neq z \wedge \mathbf{V}(\pi_1 z) \cap \mathbf{V}(\pi_2) = \varnothing$ in contradiction with the existence of an internal cycle $a\pi''\pi_2' a$ requiring $\pi_1 z = \pi' a\pi''$ and $\pi_2 = \pi_2' a\pi_2''$ so $a \in \mathbf{V}(\pi' a\pi'') \cap \mathbf{V}(\pi_2' a\pi_2'') = \mathbf{V}(\pi_1 z) \cap \mathbf{V}(\pi_2) \neq \varnothing$.

$\square$

We have the following fixpoint characterization of the elementary paths of a graph (converging in finitely many iterations for graphs without infinite paths).

Theorem 6 (Fixpoint characterization of the elementary paths of a graph) *Let $G = \langle V, E\rangle$ be a graph. The elementary paths between any two vertices of G are $\mathsf{p}^{\ni} \triangleq \alpha^{\circ\!\!-\!\!\circ} \circ \alpha^{\ni}(\Pi(G))$ such that*

$$
\begin{aligned}
\mathsf{p}^{\ni} &= \mathsf{lfp}^{\dot{\subseteq}}\, \overrightarrow{\mathscr{L}}^{\ni}_{\Pi}, & \overrightarrow{\mathscr{L}}^{\ni}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \dot{E} && \text{(Th.6.a)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}\, \overleftarrow{\mathscr{R}}^{\ni}_{\Pi}, & \overleftarrow{\mathscr{R}}^{\ni}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathbin{\dot{\cup}} \dot{E} \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p} && \text{(Th.6.b)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}\, \overleftrightarrow{\mathscr{B}}^{\ni}_{\Pi}, & \overleftrightarrow{\mathscr{B}}^{\ni}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p} && \text{(Th.6.c)}\\
&= \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\, \widehat{\mathscr{P}}^{\ni}_{\Pi}, & \widehat{\mathscr{P}}^{\ni}_{\Pi}(\mathsf{p}) &\triangleq \mathsf{p} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p} && \text{(Th.6.d)}
\end{aligned}
$$

where $\dot{E} \triangleq \lambda\, x, y \bullet (E \cap \{\langle x,\ y\rangle\})$ *in Th. 4 and* $\mathsf{p}_1 \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p}_2 \triangleq \lambda\, x, y \bullet \bigcup_{z\in V}\{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\}$. $\square$

The definition of $\mathsf{p}^{\ni}$ in Th. 6 is left-recursive in case (a), right recursive in case (b), bidirectional in case (c), and on paths only in case (d).

Proof (of Th. 6) We apply Th. 3 with abstraction $\dot{\alpha}^{\ni} \circ \alpha^{\circ\!\!-\!\!\circ}$ so that we have to abstract the transformers in Th. 4 using an exact fixpoint abstraction of Cor. 1. The commutation condition yields the transformers by calculational design.

$$
\begin{aligned}
&\dot{\alpha}^{\ni}(\mathsf{p}_1 \mathbin{\ddot{\circledcirc}} \mathsf{p}_2)\\
=\ &\dot{\alpha}^{\ni}(\lambda\, x, y \bullet \bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)) && \text{(def. } \ddot{\circledcirc} \text{ in Th. 4)}\\
=\ &\lambda\, x, y \bullet \alpha^{\ni}(\bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)) && \text{(pointwise def. } \dot{\alpha}^{\ni}\text{)}\\
=\ &\lambda\, x, y \bullet \bigcup_{z\in V} \alpha^{\ni}(\mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y))\\
& && \text{(join preservation of the abstraction in a Galois connection)}
\end{aligned}
$$

$= \lambda x, y \bullet \bigcup_{z \in V} \alpha^{\partial}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y)\})$ ⟅def. (2) of ⊚ and (5) of ⊙⟆

$= \lambda x, y \bullet \bigcup_{z \in V} \alpha^{\partial}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\partial}(\mathsf{p}_1(x, z)) \wedge \pi_2 \in \alpha^{\partial}(\mathsf{p}_2(z, y))\})$

⟅since if π_1 or π_2 are not elementary so is their concatenation $\pi_1 \odot \pi_2$⟆

$= \lambda x, y \bullet \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\partial}(\mathsf{p}_1(x, z)) \wedge \pi_2 \in \alpha^{\partial}(\mathsf{p}_2(z, y)) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\}$

⟅since, by Lem. 2, π_1 and π_2 being elementary, their concatenation $\pi_1 \odot \pi_2$ is elementary if and only if $\textsf{elem-conc?}(\pi_1, \pi_2)$ is true⟆

$= \lambda x, y \bullet \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \dot{\alpha}^{\partial}(\mathsf{p}_1)(x, z) \wedge \pi_2 \in \dot{\alpha}^{\partial}(\mathsf{p}_2)(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\}$

⟅pointwise def. $\dot{\alpha}^{\partial}$⟆

$= \dot{\alpha}^{\partial}(\mathsf{p}_1) \mathbin{\dot{⊚}^{\partial}} \dot{\alpha}^{\partial}(\mathsf{p}_2)$ ⟅def. $\dot{⊚}^{\partial}$ in Th. 6⟆ □

11 Calculational design of the elementary paths between vertices of finite graphs

In finite graphs $G = \langle V, E\rangle$ with $|V| = n > 0$ vertices, elementary paths in G are of length at most $n+1$ (for a cycle that would go through all vertices of the graph). This ensures that the fixpoint iterates in Th. 6 starting from $\dot{\varnothing}$ do converge in at most $n + 2$ iterates.

Moreover, if $V = \{z_1 \ldots z_n\}$ is finite, then the elementary paths of the $k + 2^{\text{nd}}$ iterate can be restricted to $\{z_1, \ldots, z_k\}$. This yields an iterative algorithm by application of the exact iterates multi-abstraction Th. 1 with[1]

$$
\begin{aligned}
&\alpha_0^{\partial}(\mathsf{p}) \triangleq \mathsf{p} && && (9)\\
&\alpha_k^{\partial}(\mathsf{p}) \triangleq \lambda x, y \bullet \{\pi \in \mathsf{p}(x, y) \mid \mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_k\} \cup \{x, y\}\}, && k \in [1, n] &&\\
&\alpha_k^{\partial}(\mathsf{p}) \triangleq \mathsf{p}, && k > n &&
\end{aligned}
$$

By the exclusion abstraction and pointwise extension, these are Galois connections

$$
\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xrightleftharpoons[\alpha_k^{\partial}]{\gamma_k^{\partial}} \langle V \times V \to \bigcup_{j=2}^{n+1} V^j,\ \dot{\subseteq}\rangle. \tag{10}
$$

with $\gamma_k^{\partial}(\mathsf{p}) \triangleq \mathsf{p}$ for $k = 0$ or $k > n$ and $\gamma_k^{\partial}(\mathsf{p}) \triangleq \mathsf{p} \mathbin{\dot{\cup}} \lambda x, y \bullet \{x\pi y \mid \mathbf{V}(x\pi y) \nsubseteq \{z_1, \ldots, z_k\} \cup \{x, y\}\}$ when $k \in [1, n]$.

Applying Th. 1, we get the following iterative characterization of the elementary paths of a finite graph. Notice that $\dot{⊚}_z$ in (Th.7.a) and (Th.7.b) does not require to test that the concatenation of two elementary paths is elementary while $\dot{⊚}_z^{\partial}$ in (Th.7.c) and (Th.7.d) definitely does (since the concatenated elementary paths may have vertices in common). Notice also that the iteration $\langle \vec{\mathscr{L}}^{\partial}{}_{\pi}{}^{k},\ k \in [0, n+2]\rangle$ in (Th.7.a) is not the same as the iterates $\langle \vec{\mathscr{L}}^{\partial}{}_{\Pi}{}^{k}(\dot{\varnothing}),\ k \in \mathbb{N}\rangle$

[1] This is for case (Th.7.d). For cases (a–c), we also have $\alpha_1^{\partial}(\mathsf{p}) \triangleq \mathsf{p}$ while the second definition is for $k \in [2, n+2]$.

of $\overrightarrow{\mathscr{L}}^{\backepsilon}_{\Pi}$ from $\dot{\varnothing}$, since using $\dot{\circledcirc}_{z}$ or $\dot{\circledcirc}^{\backepsilon}_{z}$ instead of $\dot{\circledcirc}^{\backepsilon}$ is the key to efficiency. This is also the case for (Th.7.b—d).

Theorem 7 ***(Iterative characterization of the elementary paths of a finite graph)*** *Let* $G = \langle V,\ E\rangle$ *be a finite graph with* $V = \{z_1, \ldots, z_n\}$, $n > 0$. *Then*

$$\mathsf{p}^{\backepsilon} = \mathsf{lfp}^{\dot{\subseteq}}\, \overrightarrow{\mathscr{L}}^{\backepsilon}_{\Pi} = \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{n+2} \quad \textit{where} \tag{Th.7.a}$$

$$\overrightarrow{\mathscr{L}}^{\backepsilon}_{\Pi}(\mathsf{p}) \triangleq \dot{E} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\backepsilon}} \dot{E} \textit{ in (Th.6.a)} \quad \textit{and} \quad \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{0} \triangleq \dot{\varnothing}, \quad \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{1} \triangleq \dot{E},$$
$$\overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{k+2} \triangleq \dot{E} \mathbin{\dot{\cup}} \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{k+1} \mathbin{\dot{\circledcirc}_{z_{k+1}}} \dot{E}, \quad k \in [0, n], \quad \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{k+1} = \overrightarrow{\mathscr{L}}^{\backepsilon}{}_{\pi}^{k}, \quad k \geqslant n+2$$

$$= \mathsf{lfp}^{\dot{\subseteq}}\, \overleftarrow{\mathscr{R}}^{\backepsilon}_{\Pi} = \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{n+2} \quad \textit{where} \tag{Th.7.b}$$

$$\overleftarrow{\mathscr{R}}^{\backepsilon}_{\Pi}(\mathsf{p}) \triangleq \dot{E} \mathbin{\dot{\cup}} \dot{E} \mathbin{\dot{\circledcirc}^{\backepsilon}} \mathsf{p} \textit{ in (Th.6.b)} \quad \textit{and} \quad \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{0} \triangleq \dot{\varnothing}, \quad \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{0} \triangleq \dot{E},$$
$$\overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{k+2} \triangleq \dot{E} \mathbin{\dot{\cup}} \dot{E} \mathbin{\dot{\circledcirc}_{z_{k+1}}} \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{k+1}, \quad k \in [0, n], \quad \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{k+1} = \overleftarrow{\mathscr{R}}^{\backepsilon}{}_{\pi}^{k}, \quad k \geqslant n+2$$

$$= \mathsf{lfp}^{\dot{\subseteq}}\, \overleftrightarrow{\mathscr{B}}^{\backepsilon}_{\Pi} = \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{n+2} \quad \textit{where} \tag{Th.7.c}$$

$$\overleftrightarrow{\mathscr{B}}^{\backepsilon}_{\Pi}(\mathsf{p}) \triangleq \dot{E} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\backepsilon}} \mathsf{p} \textit{ in (Th.6.c)} \quad \textit{and} \quad \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{0} \triangleq \dot{\varnothing}, \quad \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{1} \triangleq \dot{E},$$
$$\overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{k+2} \triangleq \dot{E} \mathbin{\dot{\cup}} \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{k+1} \mathbin{\dot{\circledcirc}^{\backepsilon}_{z_{k+1}}} \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{k+1},\ k \in [0, n],\ \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{k+1} = \overleftrightarrow{\mathscr{B}}^{\backepsilon}{}_{\pi}^{k},\ k \geqslant n+2$$

$$= \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\, \widehat{\mathscr{P}}^{\backepsilon}_{\Pi} = \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{n+1} \quad \textit{where} \tag{Th.7.d}$$

$$\widehat{\mathscr{P}}^{\backepsilon}_{\Pi}(\mathsf{p}) \triangleq \mathsf{p} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\dot{\circledcirc}^{\backepsilon}} \mathsf{p} \textit{ in (Th.6.d)},\ \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{0} \triangleq \dot{E},$$
$$\widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k+1} \triangleq \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k} \mathbin{\dot{\cup}} \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k} \mathbin{\dot{\circledcirc}^{\backepsilon}_{z_{k+1}}} \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k},\ k \in [0, n],\ \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k+1} = \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{k},\ k \geqslant n+2$$

$$\mathsf{p}_1 \mathbin{\dot{\circledcirc}_{z}} \mathsf{p}_2 \triangleq \lambda\, x, y \bullet \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge z \notin \{x, y\}\},\ \textit{and}$$
$$\mathsf{p}_1 \mathbin{\dot{\circledcirc}^{\backepsilon}_{z}} \mathsf{p}_2 \triangleq \lambda\, x, y \bullet \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\}.$$

□

Proof (of Th. 7) — The proofs in cases (Th.7.c) and (Th.7.d) are similar. Let us consider (Th.7.d). Assume $V = \{z_1 \ldots z_n\}$ and let $\widehat{\mathscr{P}}^{\backepsilon}{}_{\Pi}^{k+1} = \widehat{\mathscr{P}}^{\backepsilon}_{\Pi}(\widehat{\mathscr{P}}^{\backepsilon}{}_{\Pi}^{k})$ be the iterates of $\widehat{\mathscr{P}}^{\backepsilon}_{\Pi}$ from $\widehat{\mathscr{P}}^{\backepsilon}{}_{\Pi}^{0} = \dot{E}$ in (Th.6.d). To apply Th. 1, we consider the concrete cpo $\langle \mathcal{C},\ \dot{\subseteq}\rangle$ and the abstract cpo $\langle \mathcal{A},\ \dot{\subseteq}\rangle$ to be $\langle \mathcal{C},\ \dot{\subseteq},\ \dot{E},\ \dot{\bigcup}\rangle$ with $\mathcal{C} \triangleq x \in V \times y \in V \rightarrow \{x\pi y \mid x\pi y \in E \cup \bigcup_{k=3}^{n+1} V^k \wedge x\pi y \text{ is elementary}\}$, and the functions $\widehat{\mathscr{P}}^{\backepsilon}_{\pi k}(X) \triangleq X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}^{\backepsilon}_{z_{k+1}}} X$, $k \in [1, n]$, and $\widehat{\mathscr{P}}^{\backepsilon}_{\pi k}(X) \triangleq X$, $k = 0$ or $k > n$ which iterates from the infimum $\dot{E}$ are precisely $\langle \widehat{\mathscr{P}}^{\backepsilon}{}_{\Pi}^{k},\ i \in \mathbb{Z} \cup \{\omega\}\rangle$ where $\widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{\omega} = \dot{\bigcup}_{i \in \mathbb{Z}} \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{i} = \widehat{\mathscr{P}}^{\backepsilon}{}_{\pi}^{n+1} = \widehat{\mathscr{P}}^{\backepsilon}{}_{\Pi}^{k}$, $k > n$.

- For the infimum $\widehat{\mathscr{P}}^{\backepsilon}_{\pi 0} = \dot{E}$ the paths $x\pi y \in \dot{E}(x, y)$ of G which are elementary and have all intermediate states of π in $\varnothing = \{z_1, \ldots, z_0\}$ since π is empty.

- For the commutation, the case $k > n$ is trivial. Otherwise let $X \in \mathcal{A}$ so $x\pi y \in X(x, y)$ is elementary and has all states of π in $\{z_1, \ldots, z_k\}$

$$\begin{aligned} & \alpha^{\backepsilon}_{k+1}(\widehat{\mathscr{P}}^{\backepsilon}_{\Pi}(X)) \\ = \ & \alpha^{\backepsilon}_{k+1}(X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}^{\backepsilon}} X) && \text{(def. (Th.6.d) of } \widehat{\mathscr{P}}^{\backepsilon}_{\Pi}\text{)} \end{aligned}$$

$= \ \alpha^{\partial}_{k+1}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k+1}(X \mathbin{\dot{\circledcirc}^{\partial}} X)$ $\quad$ ⟅α^{∂}_{k+1} preserves joins in (10)⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k+1}(X \mathbin{\dot{\circledcirc}^{\partial}} X)$

⟅def. (9) of α^{∂}_{k+1} and hypothesis that all paths in X have all intermediate states in $\{z_1, \ldots, z_k\}$⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \{\pi \in X \mathbin{\dot{\circledcirc}^{\partial}} X(x, y) \mid \mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}\}$ $\quad$ ⟅def. α^{∂}_{k+1} in (9)⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \{\pi \in \bigcup_{z\in V}\{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in X(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\} \mid \mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}\}$ $\quad$ ⟅def. $\dot{\circledcirc}^{\partial}$ in Th. 6⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \bigcup_{z\in V}\{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in X(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2) \wedge \mathbf{V}(\pi_1 \odot \pi_2) \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}\}$ $\quad$ ⟅def. $\in$⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \bigcup_{z\in V}\{x\pi_1 z\pi_2 y \mid x\pi_1 z \in X(x, z) \wedge z\pi_2 y \in X(z, y) \wedge \textsf{elem-conc?}(x\pi_1 z, z\pi_2 y) \wedge \mathbf{V}(\pi_1) \cup \mathbf{V}(\pi_2) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}\}$ $\quad$ ⟅def. $\odot$, $\mathbf{V}$, and ind. hyp.⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \{x\pi_1 z_{k+1}\pi_2 y \mid x\pi_1 z_{k+1} \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge z_{k+1}\pi_2 y \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge \textsf{elem-conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)\}$

⟅($\supseteq$) follows from taking $z = z_{k+1}$; (11)
($\subseteq$) For $z \in \{z_1, \ldots, z_k\}$, the paths in $\alpha^{\partial}_{k}(X)$ are elementary through $\{z_1, \ldots, z_k\}$, so if there exist paths $x\pi_1 z \in X(x, z)$ and $z\pi_2 y \in X(z, y)$ then either $x\pi_1 z\pi_2 x$ is also elementary through $\{z_1, \ldots, z_k\}$ and already therefore belongs to $\alpha^{\partial}_{k}(X)$ or it is not elementary and then does not pass the test $\textsf{elem-conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)$;
Otherwise, if $z \in \{z_{k+2}, \ldots, z_n\}$, then the path $x\pi_1 z_{k+1}\pi_2 y$ is eliminated by $\mathbf{V}(\pi_1) \cup \mathbf{V}(\pi_2) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}$;
Finally, the only possibility is $z = z_{k+1}$, in which case all paths have the form $x\pi_1 z_{k+1}\pi_2 y$, are elementary, and with $\mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+1}\}$, as required by the def. of $\mathcal{A}$. It also holds for $\alpha^{\partial}_{k}(X)$ which is equal to $\alpha^{\partial}_{k+1}(X)$.
⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \{x\pi_1 z_{k+1} \odot z_{k+1}\pi_2 y \mid x\pi_1 z_{k+1} \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge z_{k+1}\pi_2 y \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge \textsf{elem-conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)\}$ $\quad$ ⟅def. $\odot$⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \lambda\, x, y \bullet \{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge \pi_2 \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\}$

⟅by ind. hyp. all paths in $X(x, y)$ have the form $x\pi y$⟆

$= \ \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k}(X) \mathbin{\dot{\circledcirc}^{\partial}_{z_{k+1}}} \alpha^{\partial}_{k}(X)$ $\quad$ ⟅def. $\dot{\circledcirc}^{\partial}_{z_{k+1}}$ in (Th.7.d)⟆

$= \ \widehat{\mathscr{P}^{\partial}}_{\pi k}(\alpha^{\partial}_{k}(X))$ $\quad$ ⟅(Th.7.d)⟆

We conclude by Th. 1.

— In cases (Th.7.a) and (Th.7.b), $\dot{\circledcirc}^{\partial}_{z_{k+1}}$ can be replaced by $\dot{\circledcirc}_{z_{k+1}}$ since in these cases the paths are elementary by construction. To see this, observe that for (Th.7.a), the iterates $\langle \overrightarrow{\mathscr{L}}^{\partial}{}_{\pi}{}^{k}(\dot\varnothing),\ k \in \mathbb{N} \cup \{\omega\}\rangle$ are those of the functions

$\overrightarrow{\mathscr{L}}^{\partial}{}_{\pi 0}(X) \triangleq \dot{\varnothing}$, $\overrightarrow{\mathscr{L}}^{\partial}{}_{\pi 1}(X) \triangleq \dot{E}$, and $\overrightarrow{\mathscr{L}}^{\partial}{}_{\pi k}(X) \triangleq \dot{E} \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}}^{\partial}_{z_{k-1}} \dot{E}$, $k \in [2, n+2]$, and $\overrightarrow{\mathscr{L}}^{\partial}{}_{\pi k}(X) \triangleq X$, $k > n$, so that we can consider the iterates from 1 to apply Th. 1.

- By (Th.6.a), the initialization is $\overrightarrow{\mathscr{L}}^{\partial}{}_{\Pi}(\dot{\varnothing}) \triangleq \dot{E} \mathbin{\dot{\cup}} \dot{\varnothing} \mathbin{\dot{\circledcirc}}^{\partial} \dot{E} = \dot{E}$ such that all paths $x\pi y$ in $\dot{E}(x, y)$ are elementary with π empty so $\mathbf{V}(\pi) \subseteq \varnothing = \{z_1, \ldots, z_0\}$.

- For the commutation, let $X \in \mathcal{A}$ such that all $x\pi y \in X(x, y)$ are elementary and have all states of π in $\{z_1, \ldots, z_k\}$. Then

$$
\begin{aligned}
& \alpha^{\partial}_{k+2}(\overrightarrow{\mathscr{L}}^{\partial}{}_{\Pi}(X)) && \text{\{def. iterates\}} \\
= \; & \alpha^{\partial}_{k+2}(\dot{E} \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}}^{\partial} \dot{E}) && \text{\{def. (Th.6.a) of } \overrightarrow{\mathscr{L}}^{\partial}{}_{\Pi}\text{\}} \\
= \; & \alpha^{\partial}_{k+2}(\dot{E}) \mathbin{\dot{\cup}} \alpha^{\partial}_{k+2}(X \mathbin{\dot{\circledcirc}}^{\partial} \dot{E}) && \text{\{}\alpha^{\partial}_{k+2} \text{ preserves joins in (10)\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \{\pi \in X \mathbin{\dot{\circledcirc}}^{\partial} \dot{E} \mid \mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\} && \text{\{def. } \alpha^{\partial}_{k+2} \text{ in (9)\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \{\pi \in \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\} \mid \\
& \mathbf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\} && \text{\{def. } \dot{\circledcirc}^{\partial} \text{ in Th. 6\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2) \wedge \mathbf{V}(\pi_1 \odot \\
& \pi_2) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\} && \text{\{def. } \in\text{\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \bigcup_{z \in V} \{x\pi_1 zy \mid x\pi_1 z \in X(x, z) \wedge zy \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(x\pi_1 z, zy) \wedge \\
& \mathbf{V}(\pi_1) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\} && \text{\{def. } \odot, \mathbf{V}, \dot{E} \text{ in Th. 4, and ind. hyp.\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \{x\pi_1 z_{k+2} \odot z_{k+2}\pi_2 y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge z_{k+2}\pi_2 y \in \dot{E}(z_{k+2}, y) \wedge \\
& \textsf{elem-conc?}(x\pi_1 z_{k+2}, z_{k+2}\pi_2 y)\} && \text{\{by an argument similar to (11)\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \{x\pi_1 z_{k+2} \odot z_{k+2} y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge \langle z_{k+2}, y\rangle \in E \wedge \\
& \textsf{elem-conc?}(x\pi_1 z_{k+2}, z_{k+2} y)\} && \text{\{def. (9) of } \alpha^{\partial}_{k+1} \text{ and } \dot{E} \text{ in Th. 4\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \lambda\, x, y \bullet \{x\pi_1 z_{k+2} \odot z_{k+2} y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge \langle z_{k+2}, y\rangle \in E\} \\
& \quad \text{\{since } z_{k+2} \notin \mathbf{V}(\pi_1) \text{ by induction hypothesis path so that the path } x\pi_1 z_{k+2} y \text{ is elementary\}} \\
= \; & \dot{E} \mathbin{\dot{\cup}} \alpha^{\partial}_{k+1}(X) \mathbin{\dot{\circledcirc}}_{z_{k+2}} \dot{E} && \text{\{def. } \dot{\circledcirc}^{\partial}_{z_{k+2}} \text{ in Th. 7\}} \\
= \; & \overrightarrow{\mathscr{L}}^{\partial}{}_{\pi k+2}(\alpha^{\partial}_{k+1}(X)) && \text{\{(Th.7.a)\}} \quad \square
\end{aligned}
$$

12 Calculational design of an over-approximation of the elementary paths between vertices of finite graphs

Since shortest paths are necessarily elementary, one could expect that Roy-Floyd-Warshall algorithm simply abstracts the elementary paths by their length. This is <u>not</u> the case, because the iterations in (Th.7.c) and (Th.7.d) for elementary paths are too expensive. They require to check elem-conc? in $\dot{\circledcirc}^{\partial}$ to make sure that the concatenation of elementary paths does yield an elementary path. But we can over-approximate by replacing $\dot{\circledcirc}^{\partial}$ by $\dot{\circledcirc}$ since

Lemma 3 *The length of the shortest paths in a graph is the same as the length of the shortest paths in any subset of the graph paths provided this subset contains all elementary paths.*

Proof (of Lem. 3) If $\pi_1 x \pi_2 x \pi_3$ is a non-elementary path with an internal cycle $x\pi_2 x$ of the weighted graph $\langle V, E, \boldsymbol{\omega}\rangle$ then $\pi_1 x \pi_3$ is also a path in the graph with a shorter weight, that is, by (6), $\boldsymbol{\omega}(\pi_1 x \pi_3) < \boldsymbol{\omega}(\pi_1 x \pi_2 x \pi_3)$. Since elementary paths have no internal cycles, it follows by definition of min and (7) that, for any subset P of the graph paths, we have $\boldsymbol{\omega}(P) = \boldsymbol{\omega}(P')$ whenever $\alpha^{\ni}(P) = \{\pi \in P \mid \mathsf{elem?}(\pi)\} \subseteq P' \subseteq P$. □

Corollary 2 ***(Iterative characterization of an over-approximation of the elementary paths of a finite graph)*** *Let $G = \langle V, E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then*

$$\mathsf{p}^{\ni} = \mathsf{lfp}^{\dot\subseteq}\, \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{\ni}_{\Pi} \dot\subseteq \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{n+2}_{\boldsymbol{\pi}} \tag{Cor.2.c}$$

$$\textit{where}\quad \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{0}_{\boldsymbol{\pi}} \triangleq \dot\varnothing,\quad \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{1}_{\boldsymbol{\pi}} \triangleq \dot E,\quad \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{k+2}_{\boldsymbol{\pi}} \triangleq \dot E \,\dot\cup\, \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{k+1}_{\boldsymbol{\pi}} \,\dot{\circledcirc}_{z_{k+1}}\, \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{k+1}_{\boldsymbol{\pi}}$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\, \widehat{\boldsymbol{\mathscr{P}}}^{\ni}_{\Pi} \dot\subseteq \widehat{\boldsymbol{\mathscr{P}}}^{n+1}_{\boldsymbol{\pi}} \tag{Cor.2.d}$$

$$\textit{where}\quad \widehat{\boldsymbol{\mathscr{P}}}^{0}_{\boldsymbol{\pi}} \triangleq \dot E,\quad \widehat{\boldsymbol{\mathscr{P}}}^{k+1}_{\boldsymbol{\pi}} \triangleq \widehat{\boldsymbol{\mathscr{P}}}^{k}_{\boldsymbol{\pi}} \,\dot\cup\, \widehat{\boldsymbol{\mathscr{P}}}^{k}_{\boldsymbol{\pi}} \,\dot{\circledcirc}_{z_k}\, \widehat{\boldsymbol{\mathscr{P}}}^{k}_{\boldsymbol{\pi}}$$ □

Proof (of Cor. 2) Obviously $\dot{\circledcirc}^{\ni}_{z} \dot\subseteq \dot{\circledcirc}_{z}$ so the iterates $\langle \overleftrightarrow{\boldsymbol{\mathscr{B}}}^{k}_{\boldsymbol{\pi}}, k \in [0, n+2]\rangle$ of (Cor.2.c) over-approximate those $\langle {\overleftrightarrow{\boldsymbol{\mathscr{B}}}^{\ni}_{\pi}}^{k}, k \in [0, n+2]\rangle$ of (Th.7.c). Same for (Th.7.d). □

13 The Roy-Floyd-Warshall algorithm over-approximating the elementary paths of a finite graph

The Roy-Floyd-Warshall algorithm does <u>not</u> compute elementary paths in (Th.7.d) but the over-approximation of the set of elementary paths in (Cor.2.d), thus avoiding the potentially costly test in Th. 7 that the concatenation of elementary paths in $\dot{\circledcirc}^{\ni}_{z}$ is elementary.

Algorithm 12 ***(Roy-Floyd-Warshall algorithm over-approximating the elementary paths of a finite graph)*** *Let* $G = \langle V, E\rangle$ *be a graph with* $|V| = n > 0$ *vertices. The Roy-Floyd-Warshall algorithm*

```
for x, y ∈ V do
    p(x, y) := E ∩ {⟨x, y⟩}
done;
for z ∈ V do
  for x, y ∈ V \ {z} do
    p(x, y) := p(x, y) ∪ p(x, z) ⊙ p(z, y)
  done
done
```

computes an over-approximation of all elementary paths p *of* G*.*

Proof (of Algorithm 12) The first for iteration computes $\widehat{\mathscr{P}}^0_{\boldsymbol{\pi}} \triangleq \dot{E}$ in (Cor.2.d). Then, the second for iteration should compute $\widehat{\mathscr{P}}^{k+1}_{\boldsymbol{\pi}} \triangleq \widehat{\mathscr{P}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\cup}} \widehat{\mathscr{P}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\circledcirc}_{z_k}} \widehat{\mathscr{P}}^{k}_{\boldsymbol{\pi}}$ in (Cor.2.d) since $\mathsf{p}_1 \mathbin{\dot{\circledcirc}_z} \mathsf{p}_2 = \dot{\varnothing}$ in (Th.7.d) when $z \in \{x, y\}$, in which case, $\widehat{\mathscr{P}}^{k+1}_{\boldsymbol{\pi}} = \widehat{\mathscr{P}}^{k}_{\boldsymbol{\pi}}$, which is similar to the Jacobi iterative method. However, similar to the Gauss-Seidel iteration method, we reuse the last computed $\mathsf{p}(x,z)$ and $\mathsf{p}(z, y)$, not necessarily those of the previous iterate. For the convergence of the first n iterates of the second for iteration of the algorithm, the justification is similar to the convergence for chaotic iterations [4]. ◻

14 Calculational design of the Roy-Floyd-Warshall shortest path algorithm

The shortest path algorithm of Bernard Roy [16,16,17], Bob Floyd [10], and Steve Warshall [21] for finite graphs is based on the assumption that the graph has no cycles with strictly negative weights *i.e.* $\forall x \in V \mathrel{.} \forall \pi \in \mathsf{p}(x, x) \mathrel{.} \boldsymbol{\omega}(\pi) \geqslant 0$. In that case the shortest paths are all elementary since adding a cycle of weight 0 leaves the distance unchanged while a cycle of positive weight would strictly increase the distance on the path. Otherwise, if the graph has cycles with strictly negative weights, the convergence between two vertices containing a cycle with strictly negative weights is infinite to the limit $-\infty$.

The essential consequence is that we don't have to consider all paths as in Th. 4 but instead we can consider any subset provided that it contains all elementary paths. Therefore we can base the design of the shortest path algorithm on Cor. 2. Observe that, although p may contain paths that are not elementary, d is precisely the minimal path lengths and not some strict over-approximation since

– p contains all elementary paths (so non-elementary paths are longer than the elementary path between their extremities), and

– no arc has a strictly negative weight (so path lengths are always positive and therefore the elementary paths are the shortest ones).

We derive the Roy-Floyd-Warshall algorithm by a calculation design applying Th. 1 for finite iterates to (Cor.2.d) with the abstraction $\dot{\boldsymbol{\omega}}$ (or a variant when considering (Cor.2.c)).

– for the infimum $\dot{E}$ in (Cor.2.d), we have

$$
\begin{aligned}
& \dot{\boldsymbol{\omega}}(\dot{E})\langle x,\ y\rangle \\
=\ & \boldsymbol{\omega}(\dot{E}(x, y)) && \wr\text{pointwise def. } \dot{\boldsymbol{\omega}}\int \\
=\ & \boldsymbol{\omega}(\llbracket \langle x,\ y\rangle \in E \mathrel{?} \{\langle x,\ y\rangle\} \mathrel{⦂} \varnothing \rrbracket) && \wr\text{def. } \dot{E} \text{ in Th. 4}\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(\{\langle x,\ y\rangle\}) \mathrel{⦂} \boldsymbol{\omega}(\varnothing) \rrbracket && \wr\text{def. conditional}\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \min\{\boldsymbol{\omega}(\pi) \mid \pi \in \{\langle x,\ y\rangle\}\} \mathrel{⦂} \infty \rrbracket && \wr(7)\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{⦂} \infty \rrbracket && \wr(6)\int
\end{aligned}
$$

– for the commutation with $\widehat{\boldsymbol{\mathcal{P}}}_{\pi k+1}(X) \triangleq X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}}_{z_k} X$, we have

$$
\begin{aligned}
& \dot{\boldsymbol{\omega}}(\widehat{\boldsymbol{\mathcal{P}}}_{\pi k+1}(X))\langle x,\ y\rangle \\
=\ & \dot{\boldsymbol{\omega}}(X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}}_{z_k} X)\langle x,\ y\rangle && \wr(\text{Cor.2.d})\int \\
=\ & \min(\dot{\boldsymbol{\omega}}(X)\langle x,\ y\rangle, \dot{\boldsymbol{\omega}}(X \mathbin{\dot{\circledcirc}}_{z_k} X)\langle x,\ y\rangle) \\
& && \wr\text{the abstraction } \dot{\boldsymbol{\omega}} \text{ of Galois connection (8) preserves existing joins}\int
\end{aligned}
$$

Let us evaluate

$$
\begin{aligned}
& \dot{\boldsymbol{\omega}}(X \mathbin{\dot{\circledcirc}}_{z_k} X)\langle x,\ y\rangle \\
=\ & \boldsymbol{\omega}((X \mathbin{\dot{\circledcirc}}_{z_k} X)(x, y)) && \wr\text{pointwise def. } \dot{\boldsymbol{\omega}}\int \\
=\ & \boldsymbol{\omega}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\}) && \wr\text{def. } \dot{\circledcirc}_{z_k} \text{ in Th. 7}\int \\
=\ & \min\{\boldsymbol{\omega}(\pi_1 \odot \pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\} && \wr(7)\int \\
=\ & \min\{\boldsymbol{\omega}(\pi_1) + \boldsymbol{\omega}(\pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\} && \wr\text{def. (6) of } \boldsymbol{\omega}\int \\
=\ & \llbracket z_k \in \{x, y\} \mathrel{?} \infty \mathrel{⦂} \min\{\boldsymbol{\omega}(\pi_1) \mid \pi_1 \in X(x, z_k)\} + \min\{\boldsymbol{\omega}(\pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in \\
& X(z_k, y)\} \rrbracket && \wr\text{def. min}\int \\
=\ & \llbracket z_k \in \{x, y\} \mathrel{?} \infty \mathrel{⦂} \min(\dot{\boldsymbol{\omega}}(X)(x, z_k)) + \min(\dot{\boldsymbol{\omega}}(X)(z_k, y)) \rrbracket && \wr(7) \text{ and pointwise def. } \dot{\boldsymbol{\omega}}\int
\end{aligned}
$$

so that $\dot{\boldsymbol{\omega}}(\widehat{\boldsymbol{\mathcal{P}}}_{\pi k+1}(X)) = \widehat{\boldsymbol{\mathcal{P}}}_{\delta k}(\dot{\boldsymbol{\omega}}(X))$ with $\widehat{\boldsymbol{\mathcal{P}}}_{\delta k}(X)(x, y) \triangleq \llbracket z_k \in \{x, y\} \mathrel{?} X(x, y) \mathrel{⦂} \min(X(x, y), X(x, z_k) + X(z_k, y)) \rrbracket$.

We have proved

Theorem 8 ***(Iterative characterization of the shortest path length of a graph)*** *Let* $G = \langle V, E, \boldsymbol{\omega}\rangle$ *be a finite graph with* $V = \{z_1, \ldots, z_n\}$, $n > 0$ *weighted on the totally ordered group* $\langle \mathbb{G}, \leqslant, 0, +\rangle$ *with no strictly negative weight. Then the distances between any two vertices are*

$$\mathsf{d} = \dot{\boldsymbol{\omega}}(\mathsf{p}) = \widehat{\mathscr{P}}^{n+1}_{\delta} \qquad \textit{where} \tag{Th.8}$$

$$\widehat{\mathscr{P}}^{0}_{\delta}(x, y) \triangleq [\![\langle x,\ y\rangle \in E \ ?\ \boldsymbol{\omega}(x, y) \ :\ \infty]\!],$$

$$\widehat{\mathscr{P}}^{k+1}_{\delta}(x, y) \triangleq [\![z_k \in \{x, y\} \ ?\ \widehat{\mathscr{P}}^{k}_{\delta}(x, y) \ :\ \min(\widehat{\mathscr{P}}^{k}_{\delta}(x, y), \widehat{\mathscr{P}}^{k}_{\delta}(x, z_k) + \widehat{\mathscr{P}}^{k}_{\delta}(z_k, y))]\!]$$

□

and directly get the Roy-Floyd-Warshall distances algorithm.

Algorithm 13 (Roy-Floyd-Warshall shortest distances of a graph) $G = \langle V, E, \boldsymbol{\omega}\rangle$ *be a finite graph with* $|V| = n > 0$ *vertices weighted on the totally ordered group* $\langle \mathbb{G}, \leqslant, 0, +\rangle$. *Let* $\mathsf{d} \in V \times V \to \mathbb{G} \cup \{-\infty, \infty\}$ *be computed by the Roy-Floyd-Warshall algorithm*

```
for x, y ∈ V do
    d(x, y) := if ⟨x, y⟩ ∈ E then ω(x, y) else ∞
  done;
for z ∈ V do
  for x, y ∈ V do
    d(x, y) := min(d(x, y), d(x, z) + d(z, y))
  done
done.
```

The graph has no cycle with strictly negative weight if and only if $\forall x \in V\ .\ \mathsf{d}(x, x) \geqslant 0$, *in which case* $\mathsf{d}(x, y)$ *is the length of the shortest path from* x *to* y.

Proof (of Algorithm 13) Instead of calculating the next iterate $\widehat{\mathscr{P}}^{k+1}_{\delta}$ as a function of the previous one $\widehat{\mathscr{P}}^{k}_{\delta}$ (à la Jacobi), we reuse the latest assigned values (à la Gauss-Seidel), as authorized by chaotic iterations [4]. □

15 Conclusion

We have presented a use of abstract interpretation which, instead of focusing on program semantics, focuses on algorithmics. It has been observed that graph algorithms have the same algebraic structure [3,9,11,14]. Abstract interpretation explains why.

Graph path algorithms are based on the same algebraic structure (e.g. [9, Ch. 2], [3, Table 3.1]) because they are abstractions of path finding algorithms which primitive structure $\langle \wp(V^{>1}), E, \cup, \circledcirc\rangle$ is preserved by the abstraction.

Some algorithms (*e.g.* based on (Th.6.a–b)) exactly abstract elementary paths and cycles and can therefore be designed systematically by exact fixpoint abstraction [6, theorem 7.1.0.4(3)] of the path finding fixpoint definitions. Other algorithms (such as the Roy-Floyd-Warshall or Dantzig [8] shortest path algorithms) consider fixpoint definitions of sets of paths over approximating the set of all elementary paths and cycles. We have seen for the Roy-Floyd-Warshall algorithm that the derivation of the algorithm is more complex and requires a different abstraction at each iterations (Th. 1 generalizing [6, theorem 7.1.0.4(3)]) based on a particular choice of different edges or vertices at each iteration plus chaotic iterations [4]. So from the observation of similarities, their algebraic formulation, we move to an explanation of its origin and its exploitation for the machine-checkable calculational design of algorithms.

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